Chapter 1

Continuum mechanics review

We will assume some familiarity with continuum mechanics as discussed in the context of an introductory geodynamics course; a good reference for such problems is *Turcotte and Schubert* (2002). However, here is a short and extremely simplified review of basic continuum mechanics as it pertains to the remainder of the class. You may wish to refer to our math review if notation or concepts appear unfamiliar, and consult chap. 1 of *Spiegelman* (2004) for some clean derivations.

TO BE REWRITTEN, MORE DISCUSSION ADDED.

1.1 Definitions and nomenclature

- Coordinate system. $\underline{x} = \{x, y, z\}$ or $\{x_1, x_2, x_3\}$ define points in 3D space. We will use the regular, Cartesian coordinate system throughout the class for simplicity.

  Note: Earth science problems are often easier to address when inherent symmetries are taken into account and the governing equations are cast in specialized spatial coordinate systems. Examples for such systems are polar or cylindrical systems in 2-D, and spherical in 3-D. All of those coordinate systems involve a simpler description of the actual coordinates (*e.g.* $\{r, \theta, \phi\}$ for spherical radius, co-latitude, and longitude, instead of the Cartesian $\{x, y, z\}$) that do, however, lead to more complicated derivatives (*i.e.* you cannot simply replace $\partial/\partial y$ with $\partial/\partial\theta$, for example). We will talk more about changes in coordinate systems during the discussion of finite elements, but good references for derivatives and different coordinate systems are *Malvern* (1977), *Schubert et al.* (2001), or *Dahlen and Tromp* (1998).

- Field (variable). For example $T(x, y, z)$ or $T(\underline{x})$ – temperature field – temperature varying in space.

- Indexed variables. For example, the velocity field $\underline{v}(\underline{x}) = v_i$ with $i = 1, 2, 3$ implies $\{v_1, v_2, v_3\}$, *i.e.* three variables that are functions of space $\underline{x} = \{x_1, x_2, x_3\}$.

- Repeated indices indicate summation over these components (also called Einstein summation convention).

$$\frac{\partial v_i}{\partial x_i} \quad \text{with} \quad i = 1, 2, 3 \quad \text{implies} \quad \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{1.1}$$

- In a *Eulerian frame* one uses a reference system for computations that is fixed in space, for example a computational box in which we solve for advection of temperature T in a velocity field $\underline{v}$. Local changes in, *e.g.*, temperature are then given by
$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{v}\,\underline{\nabla} T = \frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i}, \tag{1.2}$$
where D/Dt is the *total derivative* that we would experience if we were to ride on a fluid particle in the convection cell (*Lagrangian* reference frame). D/Dt takes into account local changes in a property with time (*e.g.* due to radioactive heating for T) as well as advection of temperature anomalies by means of $\underline{v}$ in and out of our local observation point.

- Tensor = indexed variable + the rule of transformation to another coordinate system.

- Traction = a force per unit area acting on a plane (a vector).

- Mean stress (= −pressure, p): $-p = \bar{\sigma} = \sigma_{ii}/3 = tr(\underline{\underline{\sigma}})/3$

- Mean strain: $\bar{\varepsilon} = \varepsilon_{ii}/3 = tr(\underline{\underline{\varepsilon}})/3 = \theta$ (also called dilatation).

- Traction/stress sign convention. Compression is negative in physics, but usually taken positive in geology. Pressure is always positive compressive.

1.2 Stress tensor

- A matrix, two indexed variables, tensor of rank two, $\underline{\underline{\sigma}}$:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \text{(2D)} \tag{1.3}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \text{(3D).} \tag{1.4}$$

- Meaning of the elements: Each row are components of the traction vectors acting on the coordinate plane normal to the respective coordinate axis, the diagonal elements are normal stresses, and off-diagonal elements are shear stresses.

  σ_{ij}: force/area (traction) on the i plane (plane with normal aligned with the i-th coordinate axis) along the j direction.

- Special properties: Symmetric, *i.e.* $\sigma_{ij} = \sigma_{ji}$. This means that only six components of σ need to be stored during computations since the other three can be readily computed. *Note*: There are different convention for the order of storing elements of σ (*e.g.* diagonal elements first, then off-diagonal; alternatively, upper right hand side ordering within, for example, a finite element program).

- Cauchy's formula: if you multiply the stress tensor (treated as a matrix) by a unit vector, n_j, which is normal to a certain plane, you will get the traction vector on this plane (see above):

$$T_{(n)i} = \underline{\underline{\sigma}}\,\underline{n} = \sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j \tag{1.5}$$

- In a model, the stress tensor is usually computed by solving the equilibrium equations.

  Note: The number of equilibrium equations is less than the number of unknown stress tensor components.

- Stress deviator

  We often decompose the stress tensor, $\underline{\underline{\sigma}}$, into a hydrostatic pressure, p, which is minus the mean stress tensor, $\bar{\sigma}$, and defined as

$$p = -\bar{\sigma} = -\frac{1}{3} tr(\underline{\underline{\sigma}}) = -\frac{\sigma_{ii}}{3} = -\frac{I_{\underline{\underline{\sigma}}}}{3}, \tag{1.6}$$

  where $I_{\underline{\underline{\sigma}}}$ is the first invariant, eq. (**??**). The deviator, or deviatoric stress, is defined as

$$\tau_{ij} = \sigma_{ij} - \delta_{ij}\bar{\sigma} = \sigma_{ij} + \delta_{ij} p. \tag{1.7}$$

  The deviatoric stress tensor invariants of $\underline{\underline{\tau}}$ are typically denoted as J (as opposed to

I for the full stress tensor, $\underline{\underline{\sigma}}$), and given by

$$J_{\underline{\underline{\tau}}} = J_1 = \tau_{ii} = 0 \tag{1.8}$$

$$JJ_{\underline{\underline{\tau}}} = J_2 = \frac{1}{2}\left(\tau_1^2 + \tau_2^2 + \tau_3^2\right) \tag{1.9}$$

$$= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{1.10}$$

$$= \frac{1}{3}I_{\underline{\underline{\sigma}}}^2 - II_{\underline{\underline{\sigma}}} \tag{1.11}$$

$$JJJ_{\underline{\underline{\tau}}} = J_3 = \tau_1\tau_2\tau_3. \tag{1.12}$$

The equivalent stress or van Mises stress is defined as

$$\sigma_e = \sqrt{3J_2} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}. \tag{1.13}$$

1.3 Strain and strain-rate tensors

- A matrix, two indexed variables, tensor of rank two, like the stress matrix.
- Meaning of the elements: Diagonal elements are elongation (rate), *i.e.* the relative changes of length in coordinate axes directions), off-diagonal elements are shears, *i.e.* deviations from 90° of the angles between lines coinciding with the coordinate axes directions before deformation.
- Special properties: symmetric.
- Strain and strain-rate tensors are a measure of the infinitesimal (small, of order %, as opposed to finite, *i.e.* large) deformation (rate). Strain and strain-rates connect to stress (forces) via the rheological (constitutive) relationships.
- Computed from the spatial gradients of displacements u and velocities v for strain and strain-rate, respectively.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.14}$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{1.15}$$

$$= \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{1.16}$$

- *Note*: The number of velocity components is smaller than the number of strain rate components.
- *Note*: Engineering strain, γ, is often used by commercial finite element packages and $\gamma = 2\varepsilon_{xy}$.

1.4 Constitutive relationships (rheology)

- A functional relationship between second rank tensors for kinematics ($\underline{\underline{\dot{\varepsilon}}}$, $\underline{\underline{\varepsilon}}$) and dynamics (forces, $\underline{\underline{\sigma}}$). For example,

  Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

  Incompressible viscous rheology: $\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{\varepsilon}_{ij}$

  Maxwell visco-elastic rheology (for deviators): $\dot{\varepsilon}_{ij} = \frac{\dot{\tilde{\sigma}}_{ij}}{2\mu} + \frac{\tilde{\sigma}_{ij}}{2\eta}$

  Here, λ, μ are elastic moduli (for an isotropic medium, there are two (bulk and shear) independent moduli which can be related to all other commonly used parameters such as Poisson's ratio). η is (dynamic, shear) viscosity, bulk viscosities are usually assumed infinite. Sometimes, kinematic viscosity $\nu = \eta/\rho$ is used.

- To solve a problem starting from the equilibrium equations for force balance, one can replace stress by strain (rate) via the constitutive law, and then replace strain (rate) by displacement (velocities). This results in a "closed" system of equations in "fundamental" variables, meaning that the number of equations is equal to the number of unknowns, the basic displacements (velocities).
- Material parameters for solid Earth problems can ideally be obtained by measuring rheology in the lab. Alternatively, indirect inferences from seismology or geodynamic modeling augmented by constraints such as post-glacial rebound need to be used.
- There are three major classes of rheologies:
  - Reversible elastic rheology at small stresses and strains over short time scales.
  - Irreversible fluid flow (creep) at large strains and over long time scales. Examples are Newtonian viscous (rate-independent) or power-law (rate/stress dependent) rheology; usually thermally activated. Intermediate stress levels.
  - Rate-independent (instantaneous), catastrophic yielding at large, limit stresses. Pressure sensitive, often temperature independent. Also called plastic, or frictional (brittle), behavior. Important for cold material over long time-scales.

1.5 Deriving a closed system of equations for a problem

1.5.1 Conservation laws

Conservation of mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0, \tag{1.17}$$

where ρ is density and $\underline{v}$ velocity. For an incompressible medium, this simplifies to

$$\underline{\nabla} \cdot \underline{v} = 0 \quad \text{or} \quad \frac{\partial v_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = 0. \tag{1.18}$$

In 2D, the incompressibility constraint can be incorporated by solving for a *stream function* (see the Lorenz problem) instead of the actual velocities. If, instead, the fundamental variables v are solved for, special care needs to be taken to ensure eq. (1.18) holds.

Conservation of momentum (equilibrium force balance)

$$\frac{D\underline{v}}{Dt} = \underline{\nabla}\, \underline{\underline{\sigma}} + \rho \underline{g}, \tag{1.19}$$

or

$$\frac{Dv_i}{Dt} = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i \tag{1.20}$$

where $\underline{g}$ is gravitational acceleration.

Conservation of energy

$$\left(\frac{\partial E}{\partial t} + v_j \frac{\partial E}{x_j} \right) + \frac{\partial q_i}{\partial x_i} = \rho Q \tag{1.21}$$

where E is energy, q_i the energy flux vector, and Q an energy source (heat production).

1.5.2 Thermodynamic relationships

Energy (heat) flux vector *vrs.* temperature gradient (Fick's law)

$$\underline{q} = -k \underline{\nabla} T \tag{1.22}$$

or

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{1.23}$$

where k is the thermal conductivity.

Equation of state 1 ("caloric" equation)

$$E = c_p \rho T \tag{1.24}$$

where c_p is heat capacity, and T is temperature. *If* all material parameters are constant (homogeneous medium), we can then write conservation of energy as

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T = \kappa \nabla^2 T + H \tag{1.25}$$

or

$$\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j} = \kappa \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} T + H = \kappa \sum_i \frac{\partial^2 T}{\partial x_i^2} + H \tag{1.26}$$

with $H = Q/\rho$ and the thermal diffusivity

$$\kappa = \frac{k}{\rho c_p}. \tag{1.27}$$

Equation of state 2: relationships for the isotropic parts of the stress/strain tensors

$$\rho = f(T, p) \tag{1.28}$$

where p is pressure (note: $\rho = \rho_0 \varepsilon_{kk}$).

Equation of state 3: Boussinesq approximation assumes the material is incompressible for all equations but the momentum equation where density anomalies are taken to be temperature dependent

$$\Delta\rho = \alpha \rho_0 \Delta T, \tag{1.29}$$

with α the thermal expansivity and $\Delta\rho$ the density difference from reference state ρ_0 for temperature difference ΔT from reference temperature T.

1.6 Summary: The general system of equations for a continuum media in the gravity field.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{1.30}$$

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{x_j} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i \tag{1.31}$$

$$\left(\frac{\partial E}{\partial t} + v_j \frac{\partial E}{x_j} \right) + \frac{\partial q_i}{\partial x_i} = \rho Q \tag{1.32}$$

$$E = c_p \rho T \tag{1.33}$$

$$\rho = f(T, P) \tag{1.34}$$

$$\tilde{\dot{\varepsilon}}_{ij} = R(\tilde{\dot{\sigma}}_{ij}, \tilde{\sigma}_{ij}) \tag{1.35}$$

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{1.36}$$

where ρ is density, v_i velocity, g_i gravitational acceleration vector, E energy, q_i heat flux vector, Q an energy source (heat production, *e.g.* by radioactive elements), c is heat capacity, T temperature, p pressure and k thermal conductivity. R indicates a general constitutive law.

Known functions, tensors and coefficients: g_i, c_p, $f(\ldots)$, ρ_0 , $R(\ldots)$, and k

Unknown functions: ρ, v_i, p, $\tilde{\sigma}_{ij}$, q_i, and T. The number of unknowns is thus equal to the number of equations.

1.6.1 Example: The Stokes system of equations for a slowly moving incompressible linear viscous (Newtonian) continuum

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{1.37}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 g_i = 0 \tag{1.38}$$

$$\rho_0 c_p \left(\frac{\partial T}{\partial t} + v_j \frac{\partial T}{x_j} \right) = \frac{\partial}{\partial x_i} \left(k \frac{\partial T}{\partial x_i} \right) + \rho_0 Q \tag{1.39}$$

$$\tilde{\dot{\varepsilon}}_{ij} = \frac{\tilde{\sigma}_{ij}}{2\eta} \tag{1.40}$$

$$\sigma_{ij} = -p\delta_{ij} + \tilde{\sigma}_{ij} \tag{1.41}$$

Major simplifications: No inertial ($D\rho/Dt$) terms (infinite Prandtl number, see non-dimensional analysis), incompressible flow, linear viscosity.

1.6.2 2D version, spelled out

Choice of coordinate system and new notation for 2D:
$g_i = \{0, -g\}$, $x_i = \{x, z\}$, $v_i = \{v_x, v_z\}$, $\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix}$ ($\sigma_{zx} = \sigma_{xz}$).
The 2D Stokes system of equations (the basis for basically every mantle convection/lithospheric deformation code):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \tag{1.42}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{1.43}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} - \rho g = 0 \tag{1.44}$$

$$\sigma_{xx} = -p + 2\eta \frac{\partial v_x}{\partial x} \tag{1.45}$$

$$\sigma_{zz} = -p + 2\eta \frac{\partial v_z}{\partial z} \tag{1.46}$$

$$\sigma_{xz} = \eta \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \tag{1.47}$$

$$\rho_0 c_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + \rho_0 Q \tag{1.48}$$

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