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THREE CYLINDERS IN CONTACT ON AN EXTENDING RUBBER SHEET: COMPARISON OF OBSERVATION AND THEORY

by

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INTRODUCTION

Aggregates dilate under stress. This property separates them from liquids and solids. The simplest approach to understanding aggregates at the microscopic level involves examining a set of like cylinders or spheres in contact. Boerner and Sclater (1992) and Boerner (1989) have examined the deformation under extension of assemblies of steel balls. To better understand the mechanical behavior of the steel balls, they suggested that the problem be reduced to that of trying to understand the motion of three cylinders in contact on an extending rubber sheet.

Sclater et al. (1990) examined analytically the behavior of three cylinders, with differing coefficients of friction, in contact on an extending rubber sheet. They found that the coefficients of friction between the cylinders themselves and between the cylinders and the rubber sheet control the behavior of the cylinders. Further, if the coefficient of friction between the cylinders themselves is less than that between the cylinders and the rubber sheet, they showed that the cylinders exhibit one of three responses which depend upon the coefficient of friction. (a) The configuration is unstable and slip of the upper cylinder and outwards rotation of the lower two cylinders occurs at the time the upper cylinder is placed on the lower two. (b) The lower two cylinders move apart incrementally but rotate inwards until slip occurs at which time they reverse direction and rotate outwards and (c) the lower two cylinders move apart rotating inwards until they are sufficiently far apart that the upper cylinder can fall into the space between the lower two.

In this paper, we carried out a series of experiments to test these predictions(Figure 1 a). Valdez et al. (1992) had already determined the coefficient of static friction between the cylinders themselves and between the cylinders and the rubber sheet. Our experiments compared the predicted extension needed for catastrophic slip to occur with that observed. In addition, we painted arrows on the flat face of the wooden dowels that we used as one set of cylinders and photographically examined the rotation of the dowels during extension.

THEORY

If three cylinders are placed on an extending rubber sheet and the coefficient of friction between the cylinders is less than that between the cylinders and the rubber sheet, Sclater et al. (1990) have shown that initially the lower two cylinders will rotate inwards. After further extension, catastrophic slip will occur when ϕ, the angle between the line between the center of the lower two cylinders and the line between the center of one of the lower cylinders and the center of the upper cylinder (Figure 1 a), decreases to a value given by the relation

$$\sin\phi = \frac{1 - \mu^2}{1 + \mu^2} \tag{1}$$

where μ is the coefficient of static friction between the two cylinders.

If β is the extended length of the rubber sheet over the original length then ϕ is related to β (Figure i b) by the following

$$\sin\phi = \frac{R\sqrt{4 - \beta^2}}{2R} \tag{2}$$

or

$$\sin\phi = \frac{\sqrt{4 - \beta^2}}{4} \tag{3}$$

combining (1) and (3) we get

$$\frac{\sqrt{4 - \beta^2}}{4} = \frac{1 - \mu^2}{1 + \mu^2} \tag{4}$$

rearranging and squaring both sides, we get

$$\frac{4 - \beta^2}{16} = \frac{(1 - \mu^2)^2}{(1 + \mu^2)^2} \tag{5}$$

solving for β

$$\beta = \left(2\sqrt{1 - \frac{\left(1 - \mu^2\right)^2}{\left(1 + \mu^2\right)^2}} \right) \tag{6}$$

Therefore

$$\beta = 2\sqrt{\frac{1 + 2\mu^2 + \mu^4 - \left(1 - 2\mu^2 + \mu^4\right)}{\left(1 + \mu^2\right)^2}} \tag{7}$$

and

$$\beta = 2\sqrt{\frac{4\mu^2}{\left(1 + \mu^2\right)^2}} \tag{8}$$

which reduces to

$$\beta = \frac{4\mu}{\left(1 + \mu^2\right)} \tag{9}$$

Equation (9) relates the extension to the coefficient of static friction and gives the theoretical amount of extension at which the upper cylinder will slip catastrophically.

Note that there is a critical value above which μ has to lie for three cylinders in contact lying on a rubber sheet to be stable. This value of μ is given by (1) when the three cylinders are in contact, i. e.,

$$\frac{\sqrt{3}}{2} = \frac{\left(1 - \mu^2\right)}{\left(1 + \mu^2\right)} \tag{10}$$

Solving (10) gives a value of μ of 0.26. When μ is below this value, slip will occur instantaneously after the top cylinder is placed on the lower two and the three cylinder in contact configuration is unstable.

EXPERIMENTAL PROCEDURE AND RESUTS

For the experiments we placed the three balls on top of each other (Figure 2 a) on a rubber sheet. We used a motor and pulley system to extend the rubber sheet until the top cylinder catastrophically slipped through the lower two cylinders which rotated out of the way. We marked the rubber with parallel lines in both the x and y direction (Figure 2 b) and measured the percent elongation in the direction of extension by measuring the distance apart of the parallel lines.

We tested seven different types of cylinder when comparing the observed and predicted extension at which catastrophic slip occurred. We examined the following: wooden dowels on wooden dowels, steel cylinders on steel cylinders, rubber covered steel cylinders on rubber covered steel cylinders, steel cylinders on rubber covered steel cylinders, Plexiglass cylinders on Plexiglass cylinders, painted Plexiglass cylinders on painted Plexiglass cylinders and Plexiglass cylinders on painted Plexiglass cylinders (Table 1). We used two different types of steel and did the experiment with the steel at two different times. The first set of steel cylinders failed catastrophically as soon as we placed the upper on the two lower cylinders. The second set actually remained stationary for a short time before slipping almost immediately after extension began (Table 1). These results are consistent with the different coefficients of friction that Valdez et al. (1992) measured for the cylinders.

We could not arrange the Plexiglass cylinders in a stable three cylinder configuration with which to start. We believe that the explanation is that the coefficient of friction of the Plexiglass is too close to the critical value for stability in the initial three cylinders in contact configuration. We were surprised by the results of the painted Plexiglass cylinder on painted Plexiglass experiment. We could not arrange the cylinders in a stable stacked position with which to start the experiment. This is the only configuration where the coefficient of friction between the cylinders themselves was greater than that between the cylinders and the rubber. A possible explanation is that this too makes the initial three cylinders in contact configuration unstable. Also, the Plexiglass cylinders on the painted Plexiglass cylinders would not stack in the initial configuration. Again the reason is unclear but it may be related to the closeness of the coefficient of friction between the cylinders themselves and that between the lower two cylinders and the rubber sheet.

We made a series of runs on each of the sets of three cylinders that we could arrange in a stable position. We compared the predicted and experimental values of β at which catastrophic failure occurs (Table 1). The agreement between the predicted amount of extension and that observed is excellent (Figure 3).

We marked the faces of the wooden dowels with arrows and photographed four of the runs with the dowels from the side. We observed the lower two dowels to rotate inwards towards each other as they started to separate (Figure 4 a-d). This is exactly what is to be predicted by the theory given the measured value for the coefficient of static friction of the the dowels. We believe that this provides further experimental validation of our theoretical analysis.

CONCLUSIONS

We observed an impressive agreement between the predicted and experimental values for β at which catastrophic slip occurred. This agreement and our ability to observe the expected inward rotation of the lower two dowels on one of the experiments provides strong validation of the appropriateness of the relation developed by Sclater et al. (1990). These experiments confirm that the behavior of three cylinders in contact on an extending rubber sheet is directly related to the coefficient of static friction between the cylinders themselves and between the cylinders and the rubber sheet.

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TABLE 1

COMPARISON OF THE PREDICTED AND OBSERVED AMOUNT OF EXTENSION REQUIRED FOR VARIOUS TYPES OF THREE CYLINDERS IN CONTACT TO FAIL BY CATASTROPHIC SLIP

Materials	Coefficient of Friction		β			
			Predicted	Experimental		
	mean	σ		n	mean	σ
Wooden dowels/ wooden dowels	0.39	0.05	1.34	8	1.27	0.07
Steel cylinders/steel cylinders #1	0.12	0.01	1.00		NA	
Steel cylinders/steel cylinders #2	0.28	0.05	1.05	5	1.07	0.02
Steel cylinders/rubber covered steel cylinders	0.59	0.04	1.75	5	1.70	0.02
Rubber covered steel/ rubber covered steel	0.72	0.05	1.90	5	1.94	0.07
Plexiglass cylinders/ Plexiglass cylinders	0.27	0.02	1.03		NA	
Plexiglass cylinders/ painted Plexiglass	0.50	0.05	1.60		NA	
Painted Plexiglass/ painted Plexiglass	0.69	0.14	1.87		NA	

FIGURE CAPTIONS

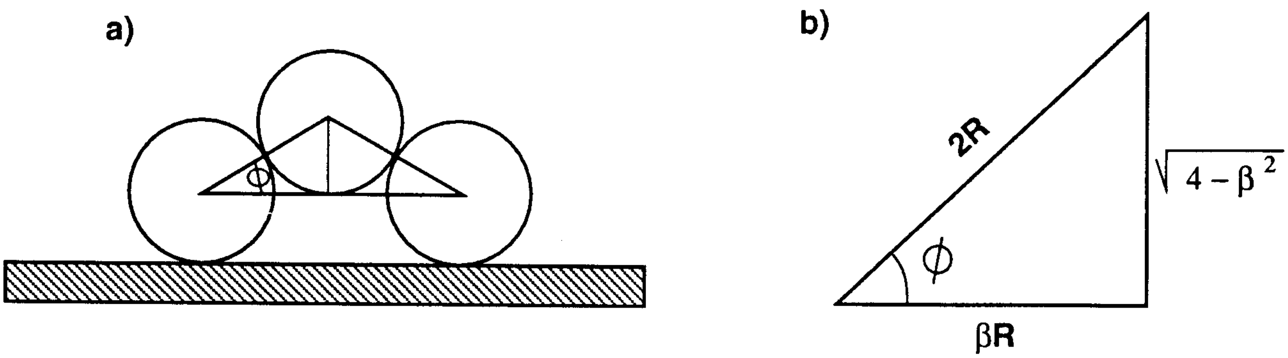
Figure 1 a A diagram of three cylinders in contact. ϕ is the angle between the line joining the center of the lower two cylinders and the line joining the center of one of the lower cylinders to the upper cylinder.

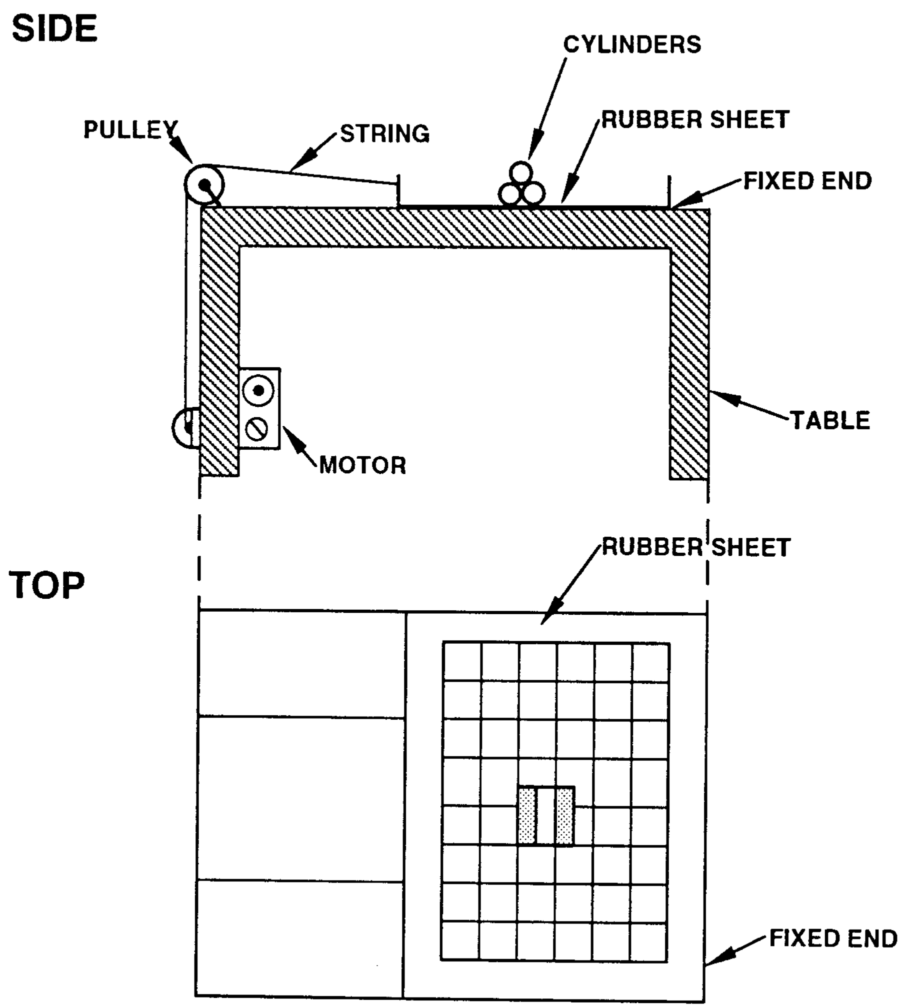
1 b The right triangle defining the relationship between ϕ and β, the amount of extension between the lower two cylinders.

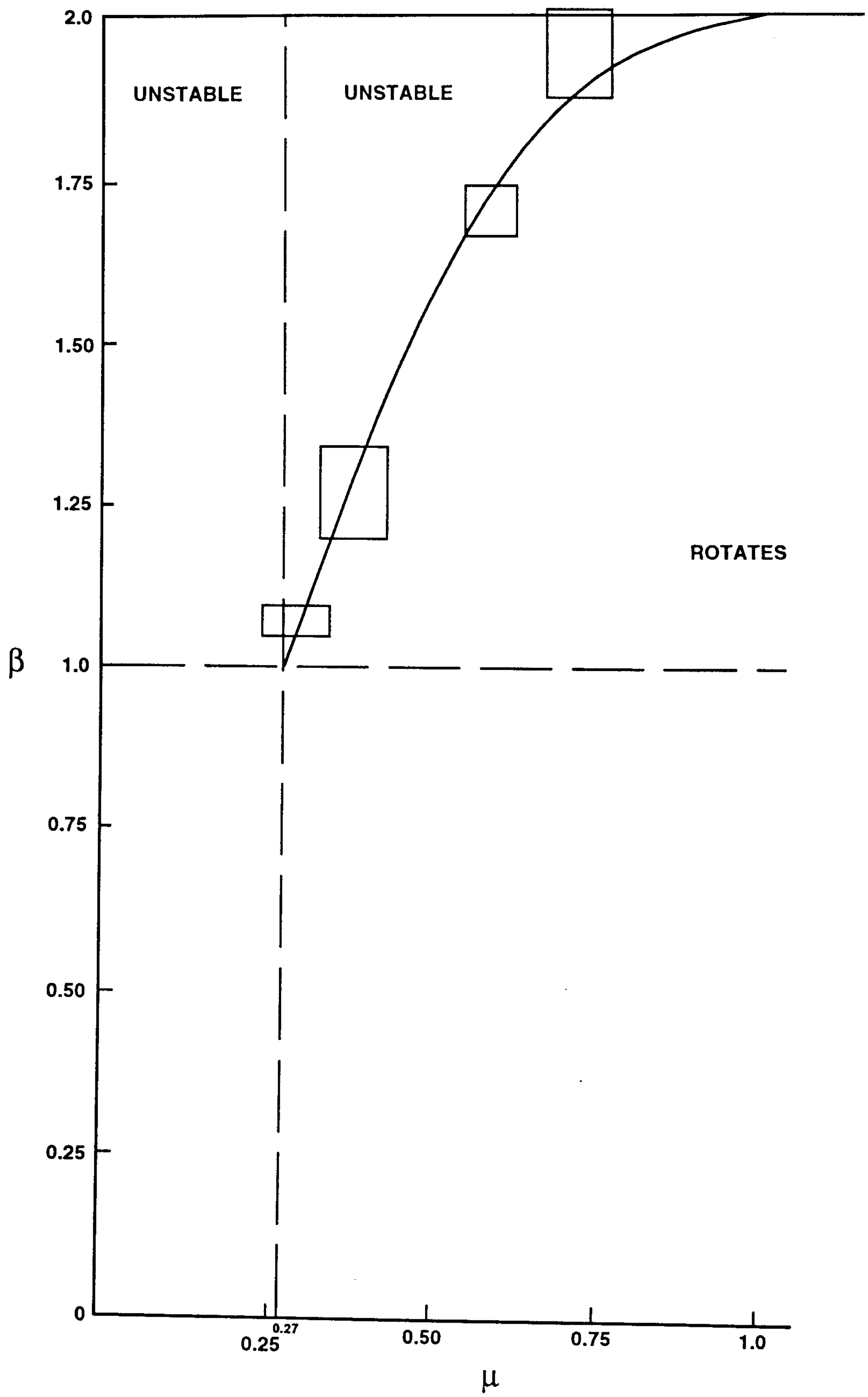
Figure 2 A side and top view of the motor, pulley and table setup used to stretch the rubber sheet. Lines were added to the rubber sheet so that the actual extension of the rubber around the cylinders could be measured

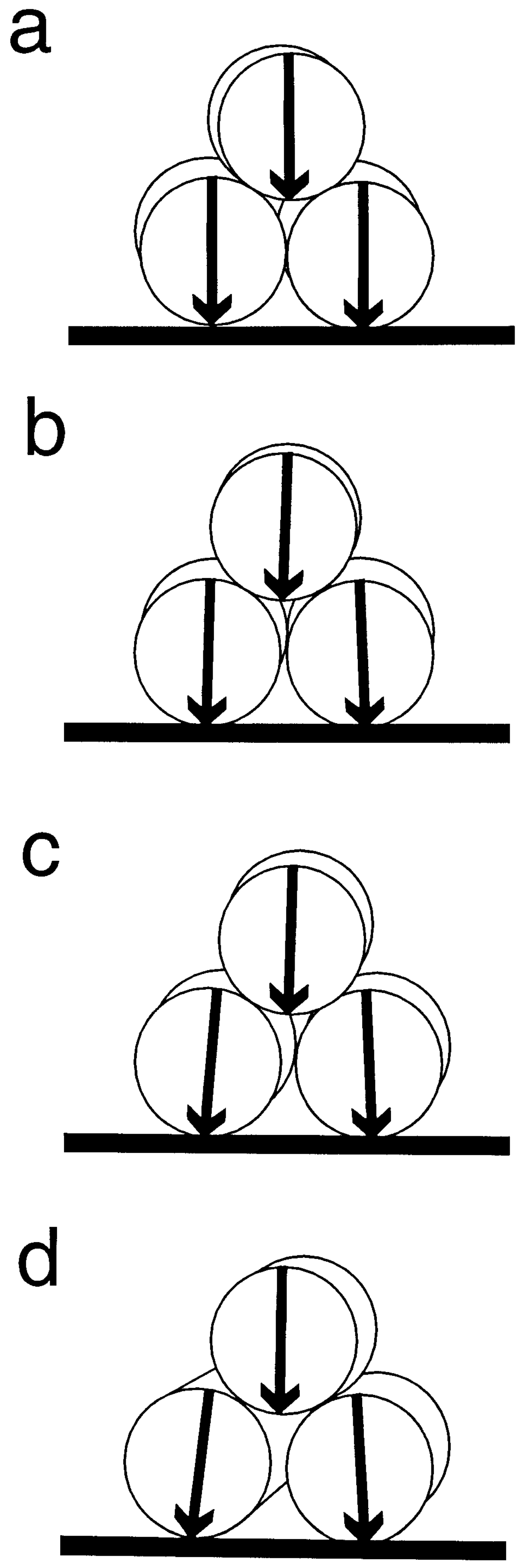
Figure 3 A plot of the extension, β, versus the coefficient of friction, μ, of the cylinders. The boxes represent the errors in both β and μ.

Figure 4 Line drawings of photographs showing the rotational configuration of three wooden dowels when β is equal to (a) 1.00, (b) 1.07, (c)1.14 and (d) 1.20. The three cylinders failed catastrophically by slip when β reached 1.29 (the camera ran out of film when β reached 1.20).


a)
φ
b)
2R
$\sqrt{4-\beta^2}$
φ
βR


SIDE
CYLINDERS
PULLEY
STRING
RUBBER SHEET
FIXED END
TABLE
MOTOR
TOP
RUBBER SHEET
FIXED END


2.0
1.75
1.50
1.25
1.0
0.75
0.50
0.25
0
β
UNSTABLE
UNSTABLE
ROTATES
0.25
0.27
0.50
0.75
1.0
μ


a
b
c
d