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Calculation of the angle of faulting of a close-packed hexagonal assembly of spheres which has undergone dilation in two and three dimensions

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INTRODUCTION

Aggregates dilate under stress. This property separates them from liquids and solids.

The simplest approach to understanding aggregates at the microscopic level involves examining a set of cylinders or spheres in contact (Jenkin, 1931; Rowe, 1962; Scott, 1963). Boerner (1989) and Boerner and Sclater (in preparation) have examined the deformation under extension of assemblies of steel balls in contact placed on an extending rubber sheet.. They have applied their results to experiments on sandboxes which geologists believe may provide analogues of the behavior of sedimentary rocks and the continental crust under extension. Our paper considers analytically the effect of dilation on faulting within a close-packed hexagonal array of balls in two and three dimensions. This dilation affects the angle of faulting and needs consideration when comparing the experimental results of Boerner (1989) and Boerner and Sclater (in preparation) with theoretical calculations.

In the experiments performed by Boerner (1989), deformation occurred by dilation and the formation of downdropping triangular fault blocks. In our treatment of both the two and three dimensional assemblies of balls, we consider first the case where all the dilation occurred only at the fault plane between a downdropping triangular fault block and the assembly of balls on either side. Then we examine the case where dilation occurs uniformly throughout the assembly of balls before concentrating on two specific fault planes to create a triangular block which then drops as a unit to create the observed faulting.

TWO DIMENSIONS

Dilation only at faults bounding blocks

Consider seven balls in the two dimensional close-packed arrangement shown in Figure 1a. If all the dilation is constrained to occur between the downdropping triangular block of three balls and the undeformed balls on either side, then the angle of faulting, α, determined by

$$\tan \alpha = z/x, \tag{1}$$

with z the vertical distance between balls 3 and 1 (2), and x the horizontal distance between balls 3 and 1 (2). The balls have radius R.

We know that all the balls are in contact, therefore

$$x^2 + z^2 = 4R^2 \tag{2}$$

and

$$x^2 = R^2 .$$

Therefore,

$$z^2 = 3R^2$$

and

$$\tan \alpha = \sqrt{3}. \tag{3}$$

Uniform dilation followed by dilation only at faults bounding blocks

Consider the situation where the balls dilate uniformly as extension takes place followed by dilation only at faults bounding triangular fault blocks. The angle of faulting will be different than that for the case where dilation occurred only at the fault bounding the blocks The new angle of faulting is a function of the geometry of the balls and the amount of extension.

Let us assume that the amount of extension is determined by a parameter β where β is the ratio of the extended to original length of the two-dimensional assembly of balls in the x direction. If we assume that just before faulting all the additional extension is concentrated at the faults bounding the blocks, then the angle of faulting, α_d, is given by

$$\tan\alpha_d = \frac{z_d}{x_d} \tag{4}$$

where $$x_d = \beta R\ .$$

Balls 2 and 3 remain in contact,

therefore $$x_d^2 + z_d^2 = 4R^2$$

or $$z_d^2 = 4R^2 - x_d^2$$

and $$z_d = R\sqrt{(4 - \beta^2)} \tag{5}$$

Substituting (5) in (4) gives

$$\tan\alpha_d = \sqrt{\left(\frac{4}{\beta^2} - 1\right)}\ . \tag{6}$$

From (6) we can see that when $\beta = 1$, $\alpha_d = \sqrt{3}$ as expected. When $\beta = 2.0$ the two lower balls are separated by two radii, and $\alpha_d = 0$.

When z_d is equal to R (Figure 1c) a new close-packed hexagonal array has been formed but with the axis rotated by 90°. In this case

$$z_d = R,$$

$$x_d = \sqrt{3}R,$$

and $$\beta = \sqrt{3}. \tag{7}$$

Substituting (7) in (6) gives

$$\tan\alpha_d = \frac{1}{\sqrt{3}}$$

or $$\alpha_d = 30°.$$

Thus, when $\alpha = 30°$ the balls return to a close-packed arrangement but stacked directly above each other (Figure 1c).

THREE DIMENSIONS

Dilation only at faults bounding triangular blocks

Consider a three dimensional close-packed hexagonal array of steel balls with the third layer of balls in the same position as the first (Figure 2a). Examined in cross section, the balls have a specific stacking plane sequence (Bollman, 1970)(Figure 2b). If motion is permitted only in the x direction and there is no dilation except at the faults bounding the triangular blocks, the faults will occur along the (0,1,1) and (0,T,1) planes. These two planes are shown as dashed lines on Figure 2b.

A suite of five balls in a close-packed hexagonal array has ball five directly above ball one (Figure 2b). Figures 3a and 3b present respectively a plan and cross-sectional view of the five balls before any dilation has taken place. We take a cartesian coordinate system whose orgin lies where balls two and three touch and where ball n lies at position (x_n, y_n, z_n). Further, we assume that all the dilation occurs along the faults bounding the triangular blocks. In this case, the angle of faulting, α_3, is given by the relation

$$\tan \alpha_3 = \frac{2z_4}{x_1} \qquad (8)$$

From simple geometry it can be shown that, if the spheres have radius R,

$$x_1 = \sqrt{3}R . \qquad (9)$$

As ball four lies in the same stacking sequence as ball one

then $$x_4^2 + z_4^2 = 4R^2$$

and $$x_4 = \frac{2}{3}x_3 = 2\frac{\sqrt{3}}{3}R \qquad (10)$$

thus $$z_4^2 = 4R^2 - \frac{4}{3}R^2$$

or $$z_4 = \sqrt{\frac{8}{3}}R \qquad (11)$$

Therefore
$$\tan \alpha_3 = 2 \frac{\sqrt{\frac{8}{3}R}}{\sqrt{3}R},$$

$$= 2\sqrt{\frac{8}{9}}, \quad (12)$$

$$\alpha_3 = 62.1°$$

Uniform dilation followed by dilation only at faults bonding blocks .

Consider five adjacent balls in the arrangement with three in one layer, the fourth in the layer above and the fifth lying in the third layer directly over the first. The position of the balls in plan view is given by Figure 3a and in cross-section by Figure 3b. Now, let the balls extend in the x direction by β. The balls will now take up the positions given by Figures 3c and d. Letting the origin of a Cartesian coordinate system lie at the point where balls one and three touch and the position of the center of ball four be x_{4d}, z_{4d} then the coordinates of the centers of the five balls are

$$\begin{aligned} 1 &: (\sqrt{3}\,R\beta,\ 0,\ 0) \\ 2 &: (0, R, 0) \\ 3 &: (0, -R, 0) \\ 4 &: (x_{4d}, 0, z_{4d}) \\ 5 &: (\sqrt{3}\,R\beta, 0, 2z_{4d}) \end{aligned} \quad (14)$$

The tangent of the angle of the fault plane is given by the height of ball five above balls two or three, $2z_{4d}$, divided by the horizontal distance between balls and balls two or three, $\sqrt{3}\ R\beta$.

Thus $$\tan\alpha_{3d} = \frac{2z_{4d}}{\sqrt{3}\ R\beta} \qquad (15)$$

Assuming that the adjacent balls in each stacking plane remain in contact and ball four remains in contact with balls two and three it is possible to determine x_{4d} and z_{4d}. The distance between balls one and four, d_{14}, is given by

$$d_{14}^2 = (\sqrt{3}R\beta - x_4{}^d)^2 + z_{4d}^2 = 4R^2 \qquad (16)$$

and the distance between balls three and four is given by

$$d_{34}^2 = x_{4d}^2 + R^2 + z_{4d}^2 = 4R^2 . \qquad (17)$$

(16) becomes

$$3R^2\beta^2 - 2\sqrt{3}R\ \beta x_{4d} + x_{4d}^2 + z_{4d}^2 = 4R^2.$$

(17) becomes

$$x_{4d}^2 + z_{4d}^2 = 3R^2.$$

Taking (17) from (16) given $$x_{4d} = \frac{3R^2\beta^2 - R^2}{2\sqrt{3}R\beta}$$

$$= \frac{R(3\beta^2 - 1)}{2\sqrt{3}\ \beta} , \qquad (18)$$

and substituting (18) in (17) gives

$$\frac{R^2}{(4)\ (3)\beta^2}(3\beta^2 - 1)^2 + z_{4d}^2 = 3R^2$$

$$R^2(9\beta^4 - 6\beta^2 + 1) + 12\beta^2 z_{4d}^2 = 36R^2\beta^2$$

$$12\beta^2 z_{4d}^2 = 36R^2\beta^2 - 9R^2\beta^4 + 6R^2\beta^2 - R^2$$

$$z_{4d}^2 = \frac{42R^2\beta^2 - 9R^2\beta^4 - R^2}{12\beta^2}$$

$$z_{4d} = R\sqrt{\left(\frac{7}{2} - \frac{3}{4}\beta^2 - \frac{1}{12\beta^2}\right)} . \qquad (19)$$

Substituting (19) in (15) gives

$$\tan\alpha_{3d} = 2R\sqrt{\frac{7}{2} - \frac{3}{4}\beta^2 - \frac{1}{12\,\beta^2}\,\frac{4}{\sqrt{3}\,B}}$$

$$= \frac{2}{\sqrt{3}\beta}\sqrt{\left(\frac{7}{2} - \frac{3}{4}\beta^2 - \frac{1}{12\beta^2}\right)} . \qquad (20)$$

At $\beta = 1$ (20) reduces to

$$\tan\alpha_{3d} = 2\sqrt{\frac{8}{9}}$$

which is the same as (12) the angle of faulting when there is no dilation.

Examining the continuing dilation of the five balls in Figures 3e and 3f, we find that ultimately, ball five rests on top of ball one and z_{4d} becomes equal to R.

From (17) when

$$z_{4d} = R$$

then,

$$x_{4d} = \sqrt{2}R ,$$

and from (18)

$$\sqrt{2} = \frac{(3\beta^2 - 1)}{2\sqrt{3}\,\beta}$$

or

$$2\sqrt{2}\,(\sqrt{3})\,\beta = 3\,\beta^2 - 1$$

$$3\beta^2 - 2\sqrt{2}\,(\sqrt{3})\,\beta - 1 = 0$$

$$3\beta^2 - \sqrt{24}\,\beta - 1 = 0$$

$$\beta = \frac{+\sqrt{24} \pm \sqrt{24 + 4(3)}}{6}$$

$$\beta = \sqrt{\frac{2}{3}} + 1 \qquad (21)$$

and from (15)

$$\tan\alpha_{3d} = \frac{2R}{\sqrt{3}\,R\left(4\sqrt{\frac{2}{3}}\right)}$$

or $$\tan\alpha_{3d} = \frac{2}{(\sqrt{3} + \sqrt{2})}$$

or $$\alpha = 32.4° .$$

At $\alpha = 32.4°$ the balls return to a close packed arrangement but rotated through 90° with the balls stacked vertically on top of each other rather than horizontally in layers (Figure 3f).

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Figure Captions

Figure 1a The position of seven spheres in a two dimensional close-packed hexagonal array with the spheres arranged in layers. The dashed lines bound a triangular array of balls which will move downwards as a unit as the rubber sheet is extended.

Figure 1b The position of the seven spheres after uniform dilation during extension. The center of sphere 5 is located at (x_d, z_d). The dashed lines bound a downdropping triangular block.

Figure 1c The position of the five spheres after enough extension to reproduce a close-packed hexagonal array but with the spheres now arranged in columns.

Figure 2a A close-packed arrangement of equal spheres showing the possible positions (A,B, and C) of successive layers of balls. DD' and EE' represent sections cut through the array of spheres. The position of spheres 1, 2, and 3 which lie in the second layer are discussed in the text.

Figure 2b A diagram showing the stacking sequence of the centers of the spheres in planes DD' (closed circles) and EE' (open circles) with the dashed lines representing the preferred planes of failure on extension and the heavy lines the projection of the unit cell in the stacking sequence. The position of spheres 1, 3, 4 and 5 are discussed in the text.

Detecting Clathrate Concentrations Through High Resolution Seismic Velocity Analysis of Shallow Sediments

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Figure 3a A plan view of a suite of five equal spheres in a close-packed hexagonal arrangement with spheres 1,2, and 3 in one layer, sphere 4 in the layer above and sphere 5 lying in the third layer directly over sphere 1. The position of spheres 1, 2 and 3 within an assembly of balls is shown on figure 2a.

Figure 3b A cross-section view of the 5 spheres. The dashed spheres represent the balls in the same plane as sphere 1 and the heavy lines those in the same plane as sphere 3. The position of spheres 1, 3, 4 and 5 are shown on a stacking sequence in figure 2b.

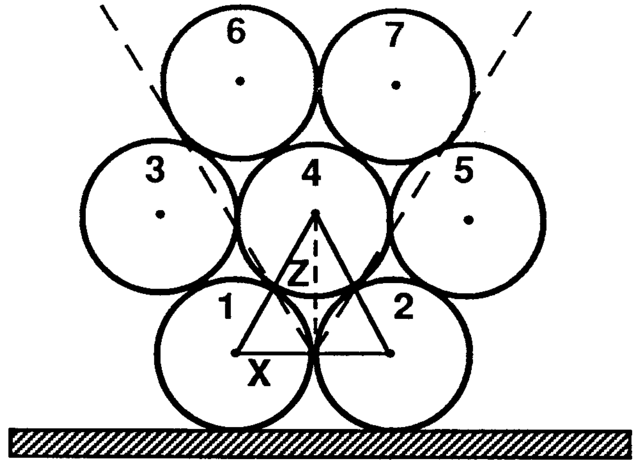
Figure 3c A plan view of the five spheres after uniform dilation.

Figure 3d A cross-section view of the five spheres after extension. The center of sphere 4 is located at (x_{4d}, 0, z_{4d}).

Figure 3e A plan view of the spheres after enough extension for sphere 5 to rest on sphere 1.

Figure 3f A cross-sectional view of the spheres in 3e. Note that this new array is also a close-packed hexagonal arrangement but with the spheres arranged in columns rather than layers.

a)



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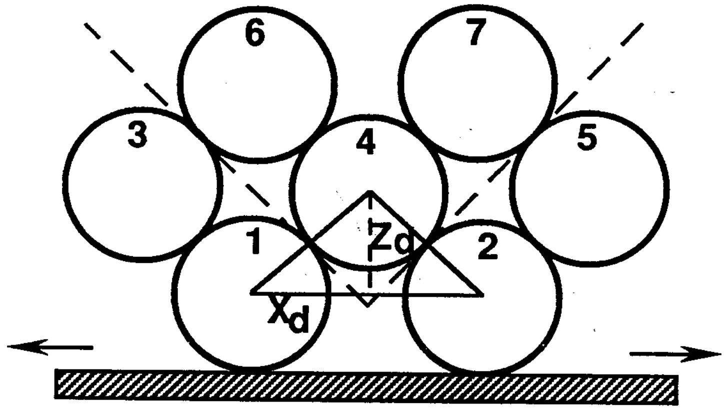

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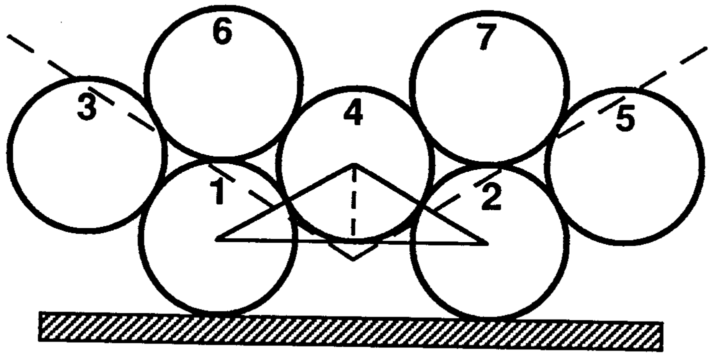

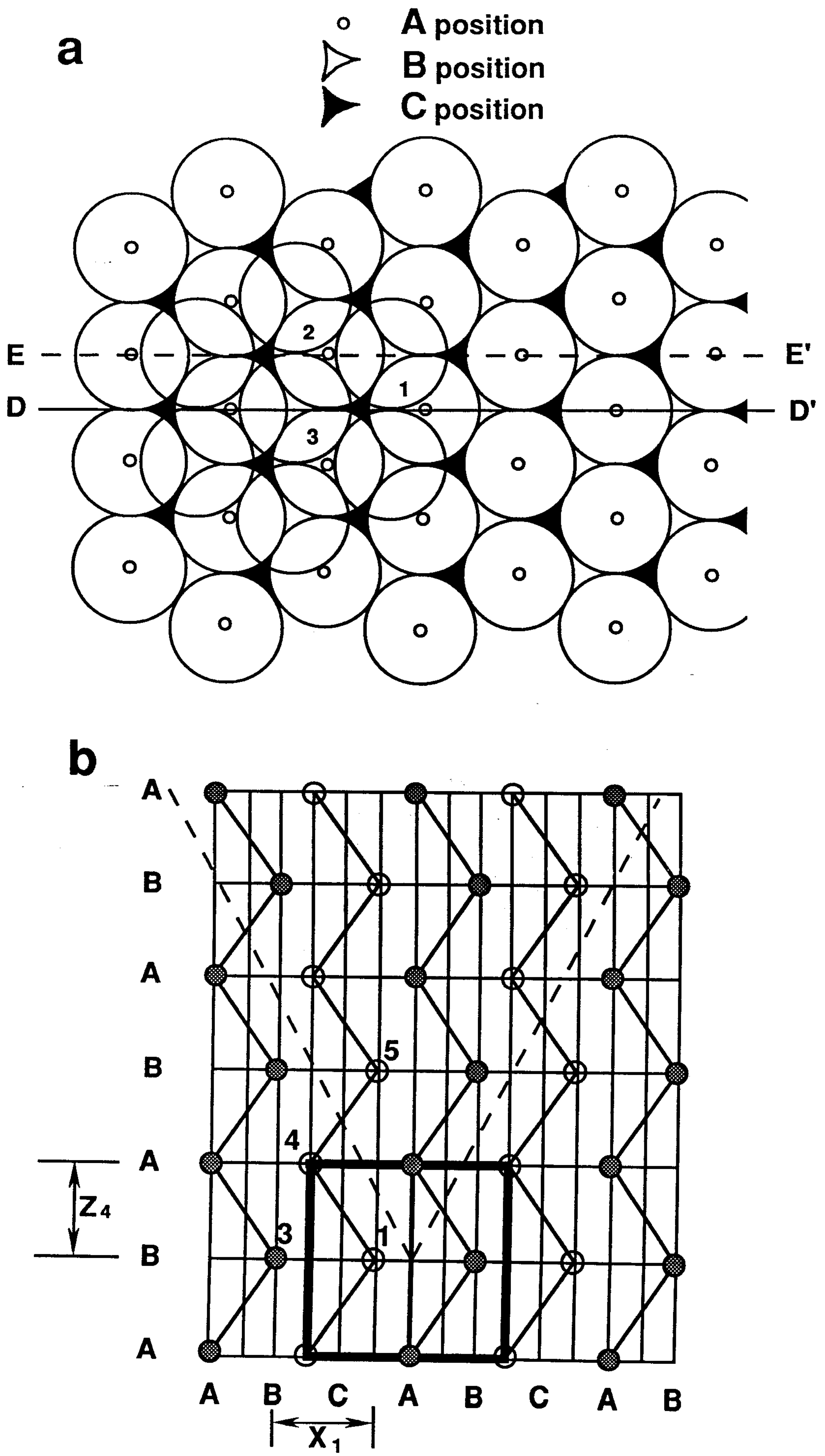
a
A position
B position
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2
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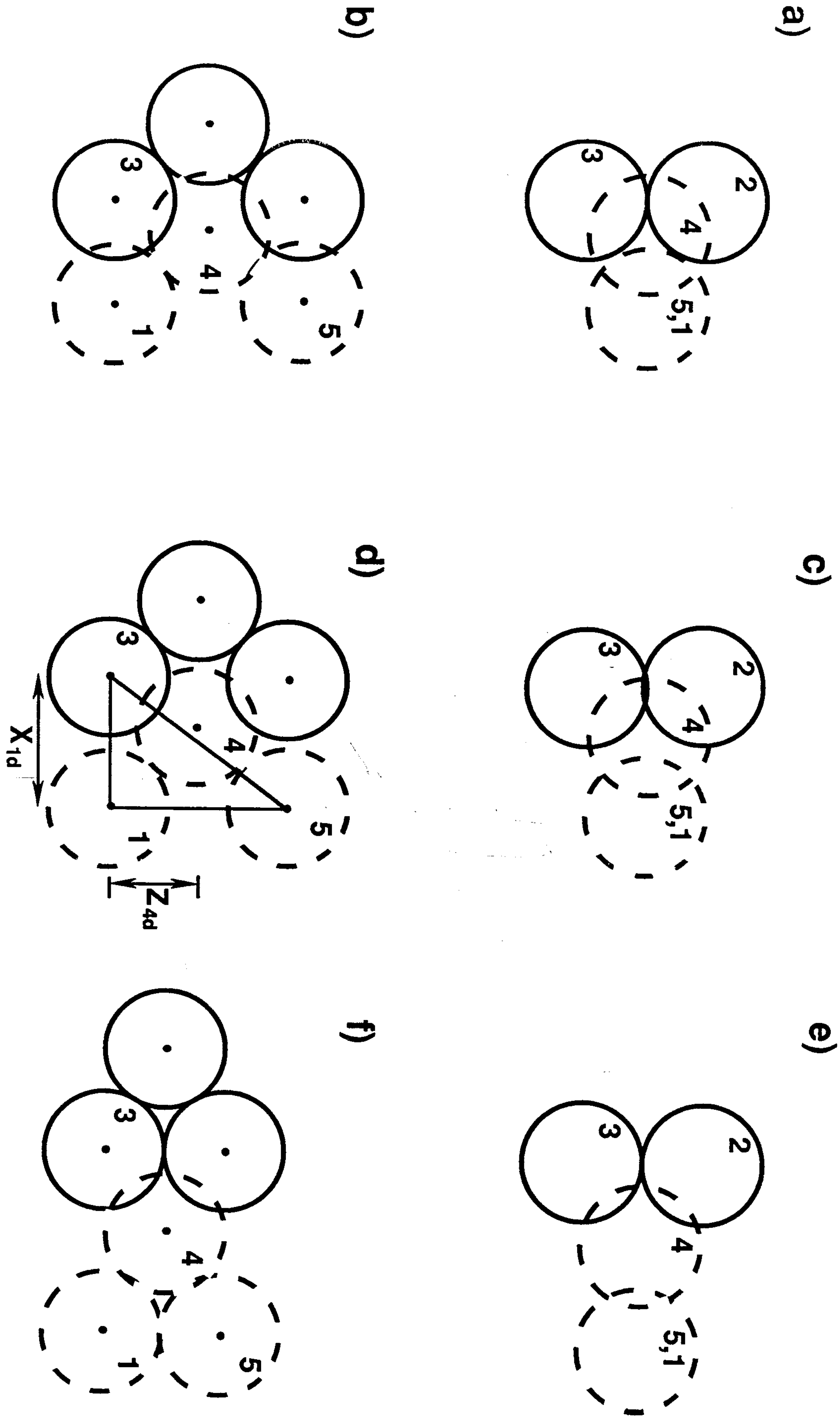
a)
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4
5,1
1
5
X_{1d}
Z_{4d}