TR 108

DERIVATION OF FORMULAS FOR THE CONVERGENCE OF 15-DEGREE CASCADED MIGRATION

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by

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The purpose of this technical report is to derive some equations contained in Cambois' paper (1989). The same notations are used.

First, let us derive equations (17), (18) and (19) from equations (12) to (16). We have

$$a_0^{n+1} = a_0^n \tag{T1}$$

$$a_i^{n+1} = a_i^n - b_{i-1}^n \quad \text{for all } i \geq 1 \tag{T2}$$

$$a_0^n b_0^n = 1 \tag{T3}$$

$$\sum_{j=0}^{i} a_j^n b_{i-j}^{n'} = 0 \quad \text{for all } i \geq 1 \tag{T4}$$

$$a_0^0 = 1 \text{ and } a_i^0 = 0 \quad \text{for all } i \geq 1 \tag{T5}$$

From (T1) we deduce that for any n, $a_0^n = a_0^0 = 1$. Then from (T3) we have $b_0^n = 1$. This proves equation (17).

Now let us consider $i \geq 1$ and let us assume that for any n and for any $j < i$, a_j^n and b_j^n are polynomials in n of degree j. If we can show that a_i^n and b_i^n are polynomials in n of degree i, then the previous assumption is a true statement (because it is true for i=0 to start with). We have:

$$a_i^{n+1} = a_i^n - b_{i-1}^n = a_i^0 - \sum_{k=0}^{n} b_{i-1}^k = - \sum_{k=0}^{n} b_{i-1}^k \tag{T6}$$

It is known that $\sum_{k=0}^{n} k^j$ is a polynomial in n of degree j+1. Thus, a_i^{n+1} and a_i^n are polynomial in n of degree i. From (T4) we have

$$b_i^n = - \sum_{j=0}^{i-1} a_j^n b_{i-j}^n \tag{T7}$$

therefore b_i^n is also a polynomial in n of degree i.

From what has just been derived we introduce the notations:

$$a_i^n = \sum_{k=0}^{i} \alpha_{ik}\, n^k \tag{T8}$$

$$b_i^n = \sum_{k=0}^{i} \beta_{ik}\, n^k \tag{T9}$$

Note that from (T5) we have $a_i^0 = 0$ then $\alpha_{i0} = 0$. If λ is defined such as:

$$\lambda_{ik} = \frac{\alpha_{ik}}{2^i} \tag{T10}$$

then equation (18) is proved.

Now let us consider $i \geq 1$ and let us assume that for $1 \leq j \leq i$ we have

$$\alpha_{jj} = \frac{2^j}{j!} \prod_{k=0}^{j-1} \left(k - \frac{1}{2}\right) \tag{T11}$$

Let us calculate $\alpha_{i+1,i+1}$. From (T6) we have

$$\alpha_{j+1,j+1} = -\frac{\beta_{jj}}{j+1} \tag{T12}$$

then for $1 \leq j \leq i\text{-}1$ we have

$$\beta_{jj} = -\frac{2^{j+1}}{j!} \prod_{k=0}^{j} \left(k - \frac{1}{2}\right) \tag{T13}$$

From (T7) we have

$$\beta_{ii} = -\sum_{k=1}^{i} \alpha_{kk}\, \beta_{i-k,i-k} \tag{T14}$$

thus

$$\beta_{ii} = \sum_{k=1}^{i} \frac{2^k}{k!} \prod_{j=0}^{k-1} \left(j - \frac{1}{2}\right) \frac{2^{i-k+1}}{(i-k)!} \prod_{j=0}^{i-k} \left(j - \frac{1}{2}\right) \tag{T15}$$

Using the notation

$$C_i^k = \frac{i!}{k!\,(i-k)!} \tag{T16}$$

we have

$$\beta_{ii} = \frac{2^{i+1}}{i!} \sum_{k=1}^{i} C_i^k \prod_{j=0}^{k-1} \left(j - \frac{1}{2}\right) \prod_{j=0}^{i-k} \left(j - \frac{1}{2}\right) \tag{T17}$$

So now, we have to prove that

$$\sum_{k=1}^{i} C_i^k \prod_{j=0}^{k-1} \left(j - \frac{1}{2}\right) \prod_{j=0}^{i-k} \left(j - \frac{1}{2}\right) = -\prod_{j=0}^{i} \left(j - \frac{1}{2}\right) \tag{T18}$$

Note that (T18) is true for i=1. Let us consider $i > 1$ and suppose that (T18) is true for all $j \leq i$. Let show that it is also true for i+1:

$$-\prod_{j=0}^{i+1} \left(j - \frac{1}{2}\right) = -\left(i + \frac{1}{2}\right) \prod_{j=0}^{i} \left(j - \frac{1}{2}\right) \tag{T19}$$

thus

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\left(i+\frac{1}{2}\right)\sum_{k=1}^{i} C_i^k \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k}\left(j-\frac{1}{2}\right) \tag{T20}$$

and

$$i+\frac{1}{2}=\left(k-\frac{1}{2}\right)+\left(i-k+\frac{1}{2}\right)+\frac{1}{2} \tag{T21}$$

then

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\sum_{k=1}^{i} C_i^k \prod_{j=0}^{k}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k}\left(j-\frac{1}{2}\right)+\sum_{k=1}^{i} C_i^k \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right)-\frac{1}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right) \tag{T22}$$

By shifting the index k in the first term of the right hand side expression of (T22) we have

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\sum_{k=2}^{i+1} C_i^{k-1} \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right)+\sum_{k=1}^{i} C_i^k \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right)-\frac{1}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right) \tag{T23}$$

or

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\sum_{k=2}^{i} \left(C_i^{k-1}+C_i^k\right) \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right)-\frac{1}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right)-\frac{i}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right)-\frac{1}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right) \tag{T24}$$

Remembering the general formula

$$C_i^{k-1}+C_i^k=C_{i+1}^k \tag{T25}$$

(T24) can be written

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\sum_{k=2}^{i} C_{i+1}^k \prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right)-\frac{(i+1)}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right)-\frac{1}{2}\prod_{j=0}^{i}\left(j-\frac{1}{2}\right) \tag{T26}$$

Furthermore we have $C_{i+1}^1=i+\frac{1}{2}$ and $C_{i+1}^{i+1}=1$. Therefore:

$$-\prod_{j=0}^{i+1}\left(j-\frac{1}{2}\right)=\sum_{k=1}^{i+1} C_{i+1}^{k}\prod_{j=0}^{k-1}\left(j-\frac{1}{2}\right)\prod_{j=0}^{i-k+1}\left(j-\frac{1}{2}\right) \tag{T27}$$

Hence, (T18) is always true. This finishes to prove equation (T11). Remembering (T10), we have derived equation (19).

Let us compute numerically some of the polynomials P, Q and R as defined in equations (20) to (22). Remembering that $b_0^n = 1$ for any n, and using equation (T6), we have

$$a_1^{n+1} = -(n+1) \tag{T28}$$

thus

$$P_1(n) = -\frac{n}{2} \tag{T29}$$

and

$$Q_1\left(\frac{1}{N}\right) = -\frac{1}{2} \tag{T30}$$

From equation (19) we get $\lambda_{11} = -\frac{1}{2}$, then

$$R_1(x) = 1 \tag{T31}$$

which proves equation (29).

Now from (T7) and (T28) we have $b_1^n = n$. Using equation (T6) we get

$$a_2^{n+1} = -\frac{n(n+1)}{2} \tag{T32}$$

thus

$$P_2(n) = -\frac{n(n-1)}{8} \tag{T33}$$

and

$$Q_2\left(\frac{1}{N}\right) = -\frac{1}{8}\left(1 - \frac{1}{N}\right) \tag{T34}$$

From equation (19) we get $\lambda_{22} = -\frac{1}{8}$, then

$$R_2(x) = 1 - x \tag{T35}$$

which proves equation (30). Equations (31) and (32) can be derived following exactly the same scheme.

Carbois

A PROOF FOR THE CONVERGENCE OF 15-DEGREE CASCADED MIGRATION

Introduction

Larner and Beasley (1987) introduced cascaded migrations and showed, without proving it rigorously, that iterating a 15-degree migration with conditions on the velocity was equivalent to a 90-degree migration. I will give here a proof based on a power series expansion that makes it possible to compare this method with other kinds of migration. The equivalent formulation can be used to cascade 45-degree migrations which results in increased accuracy for the equivalent number of iterations.

Problem formulation

Assuming constant velocity, the 15-degree equation is represented in the frequency domain by the dispersion relation (Claerbout, 1985)

$$k = \omega - \frac{v^2 k_x^2}{2\omega} \tag{1}$$

where v is half the medium velocity, k_x is the lateral wavenumber, ω is the frequency of the unmigrated data and k is the frequency of the 15-degree migrated data. Doing N stages of 15-degree migration, each one with velocity $\frac{v}{\sqrt{N}}$, the first stage is defined by

$$k_N^1 = \omega - \frac{v^2 k_x^2}{2N\omega} \tag{2}$$

the second stage by

$$k_N^2 = k_N^1 - \frac{v^2\, k_x^2}{2\, N\, k_N^1} \tag{3}$$

and the (n+1)th stage by

$$k_N^{n+1} = k_N^n - \frac{v^2\, k_x^2}{2\, N\, k_N^n} \quad 0 \leq n < N \tag{4}$$

Revising the notation so that

$$u_N^n = \frac{k_N^n}{\omega}, \text{ thus } u_N^0 = 1 \tag{5}$$

$$\alpha = \frac{v\, k_x}{\omega} \tag{6}$$

then equation (4) is equivalent to

$$u_N^{n+1} = u_N^n - \frac{\alpha^2}{2\, N} \frac{1}{u_N^n} \quad 0 \leq n < N \tag{7}$$

Note that α represents the sine of the angle of propagation, thus $|\alpha| \leq 1$. The problem is to prove that

$$u_\infty = \frac{k_\infty}{\omega} = \lim_{N \to \infty} u_N^N = \sqrt{1 - \alpha^2} \tag{8}$$

Proof

Assuming that u_N^n can be written as:

$$u_N^n = \sum_{i=0}^{\infty} a_i^n \left(\frac{\alpha^2}{2N}\right)^i \tag{9}$$

equation (7) is now equivalent to

$$\sum_{i=0}^{\infty} a_i^{n+1} \left(\frac{\alpha^2}{2N}\right)^i = \sum_{i=0}^{\infty} a_i^n \left(\frac{\alpha^2}{2N}\right)^i - \left(\frac{\alpha^2}{2N}\right) \sum_{i=0}^{\infty} b_i^n \left(\frac{\alpha^2}{2N}\right)^i \tag{10}$$

where

$$\left(\sum_{i=0}^{\infty} a_i^n x^i\right) . \left(\sum_{i=0}^{\infty} b_i^n x^i\right) = 1 \tag{11}$$

Arranging terms of the same power of $\frac{\alpha^2}{2N}$ in equation (10) we have

$$a_0^{n+1} = a_0^n \tag{12}$$

$$a_i^{n+1} = a_i^n - b_{i-1}^n \quad \text{for all } i \geq 1 \tag{13}$$

and from (11) we obtain

$$a_0^n b_0^n = 1 \tag{14}$$

$$\sum_{j=0}^{i} a_j^n b_{i-j}^n = 0 \quad \text{for all } i \geq 1 \tag{15}$$

Equations (12) to (15) represent a recursive system where the level n+1 is calculated from the levels 0 to n. The initial condition is given by equations (5) and (9) when n=0:

$$a_0^0 = 1 \text{ and } a_i^0 = 0 \quad \text{for all } i \geq 1 \tag{16}$$

Then, starting from (16) and using the system (11-15) a_N^N and u_N^N can be estimated. Unfortunately their expression is very complicated, but some results can easily be derived:

- using (12), (14) and (16) we have

$$a_0^n = b_0^n = 1 \quad \text{for } 0 \leq n \leq N \tag{17}$$

- using (13), (15) and (16) for $i \geq 1$ we have (Cambois, 1989)

$$a_i^n = 2^i \sum_{k=1}^{i} \lambda_{ik}\, n^k \tag{18}$$

and

$$\lambda_{ii} = \frac{1}{i!} \prod_{j=0}^{i-1} \left(j - \frac{1}{2}\right) \tag{19}$$

Using the notations

$$P_i(n) = \frac{a_i^n}{2^i} = \sum_{k=1}^{i} \lambda_{ik}\, n^k \tag{20}$$

$$Q_i\left(\frac{1}{N}\right) = \frac{P_i(N)}{N^i} = \sum_{k=1}^{i} \lambda_{ik} \frac{1}{N^{i-k}} \tag{21}$$

$$R_i(x) = \frac{Q_i(x)}{\lambda_{ii}} \tag{22}$$

P_i is a polynomial of degree i, Q_i and R_i are polynomials of degree i-1. We have

$$u_N^n = 1 + \sum_{i=1}^{\infty} 2^i P_i(n) \left(\frac{\alpha^2}{2N}\right)^i \tag{23}$$

hence

$$u_N^N = 1 + \sum_{i=1}^{\infty} \frac{P_i(N)}{N^i} \alpha^{2i} \tag{24}$$

or

$$u_N^N = 1 + \sum_{i=1}^{\infty} Q_i\left(\frac{1}{N}\right) \alpha^{2i} \tag{25}$$

Note that $Q_i(0) = \lambda_{ii}$ then when N goes to infinity we have

$$u_\infty = 1 + \sum_{i=1}^{\infty} \lambda_{ii}\, \alpha^{2i} \tag{26}$$

Remembering (19) we have

$$u_\infty = 1 + \sum_{i=1}^{\infty} \frac{\alpha^{2i}}{i!} \prod_{j=0}^{i-1} \left(j - \frac{1}{2}\right) = \sqrt{1 - \alpha^2} \tag{27}$$

thus

$$k_\infty = \omega \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}} = k_\tau \tag{28}$$

There is a more elegant way of proving the same result (Diet, pers. comm.) but this method gives a power series expansion of the vertical wavenumber for any number of iteration. This allows us to make comparisons with existing migration schemes.

Comparison with other migration schemes

When N is finite, the frequency of the migrated data is represented by $k_N = \omega\, u_N^N$. Equation (26) gives a power series expansion of u_N^N and thus k_N. Unfortunately, I did not find a general expression for Q_i (or R_i) but it can easily be computed when one knows Q_j for $j < i$, using equations (12) to (15). For instance (Cambois, 1989):

$$R_1(x) = 1 \tag{29}$$

$$R_2(x) = 1 - x \tag{30}$$

$$R_3(x) = (1 - x)^2 \tag{31}$$

$$R_4(x) = (1 - x)\left(1 - \frac{2}{3}x\right)\left(1 - \frac{7}{5}x\right) \tag{32}$$

Now if we compare the series expansions of various migrations defined by Claerbout (1985)

90° $$\frac{k}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8}\alpha^4 - \frac{1}{16}\alpha^6 - \frac{5}{128}\alpha^8 \ldots \tag{33}$$

60° $$\frac{k}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8}\alpha^4 - \frac{1}{16}\alpha^6 - \frac{1}{32}\alpha^8 \ldots \tag{34}$$

45° $$\frac{k}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8}\alpha^4 - \frac{1}{32}\alpha^6 - \frac{1}{128}\alpha^8 \ldots \tag{35}$$

15° $$\frac{k}{\omega} = 1 - \frac{1}{2}\alpha^2 \tag{36}$$

with a 15-degree migration cascaded N times

$$\frac{k_N}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8}\left(1 - \frac{1}{N}\right)\alpha^4 - \frac{1}{16}\left(1 - \frac{1}{N}\right)^2\alpha^6 - \frac{5}{128}R_4\left(\frac{1}{N}\right)\alpha^8 \ldots \tag{37}$$

we can see that after N stages, a 15-degree cascaded migration may be more accurate than a 45-degree or a 60-degree migration for steep angles of propagation, depending on the value of N. The larger the number of iterations, the better the result. This confirms Larner and Beasley's empirical findings. One can also cascade 45-degree migrations with a medium velocity $\frac{v}{\sqrt{N}}$ and find a very similar result:

$$\frac{k_N}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8}\alpha^4 - \frac{1}{16}\left(1 - \frac{1}{2N^2}\right)\alpha^6 - \frac{5}{128}\left(1 - \frac{6}{5N^2} + \frac{2}{5N^3}\right)\alpha^8 \ldots \tag{38}$$

which is more accurate than a 15-degree cascaded migration for the same number of iterations. It is also interesting to take the series expansion of Stolt migration (1978) with the "W factor":

$$\frac{k}{\omega} = 1 - \frac{1}{2}\alpha^2 - \frac{1}{8} W \alpha^4 - \frac{1}{16} W^2 \alpha^6 - \frac{5}{128} W^3 \alpha^8 \dots \qquad (39)$$

This is comparable to N stages of a 15-degree cascaded migration when Stolt's W factor is:

$$W = 1 - \frac{1}{N} \qquad (40)$$

Conclusion

This way of proving the convergence of the 15-degree cascaded migration leads to a better understanding of the scheme and can predict better results for other kinds of cascaded migration. However, it is very difficult to estimate the accuracy of each scheme in term of "degree" as long as the Q_i polynomials are not known analytically.

References

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