

1 Scaling analysis and non-dimensional numbers

While this is a textbook on numerical analysis, it is crucial to keep the nature of the physical processes which we would like to model in mind. This will help guide your judgement of what are reasonable solutions, what are artifacts, and help with the algorithmic design itself. While we cannot review all of the physics underlying the modeling examples here fully, it is very helpful to consider scaling analysis to get a feeling for the order of magnitude of likely solutions, and the importance of different terms in the equations we would like to model.

Reading

- *Spiegelman (2004)*, sec. 1.4
- *Turcotte and Schubert (2002)*, Google, and Wikipedia for reference and material parameters

1.1 Scaling analysis

Scaling analysis refers to order of magnitude estimates on how different processes work together and control a system. While this is a text on numerical analysis, such theoretical considerations are very useful if we are interested in getting a quick idea of the values that are of relevance for a problem, and for the order of magnitude for solutions. Comparing these estimates with the numerical results is always good practice and part of a basic set of plausibility checks that have to be conducted.

For example, shear stress τ (in units of Pa) for a Newtonian viscous rheology with viscosity η (in units of Pa s) is given by the simple constitutive law

$$\tau = 2\eta\dot{\epsilon} \quad (1)$$

where $\dot{\epsilon}$ is the strain-rate (in units of s^{-1}). Say, we wish to estimate the typical amplitudes of shear stress in a part of the crust that we know is being sheared at some (*e.g.* plate-) velocity u over a zone of width L . The strain-rate in 3-D is really a tensor, $\underline{\dot{\epsilon}}$, with 3×3 components that depends on the spatial derivatives of the velocity like so

$$\underline{\dot{\epsilon}} = \dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2)$$

and has to be either constrained by kinematics or inferred from the full solution. However, for our problem, we only need a “characteristic” value, *i.e.* correct up to a factor of ten or so. Strain-rate is physically the change in velocity over length, and the characteristic strain-rate is then given by

$$\dot{\epsilon} \propto \frac{v}{L} \quad (3)$$

where \propto means “proportional to”, or “scales as”, to indicate that eq. (3) is not exact. Assuming we know the viscosity η , we can then estimate the typical stress in the shear zone to be

$$\tau \propto 2\eta \frac{v}{L}. \quad (4)$$

If you think about the units of all quantities involved (“dimensional analysis”), then this scaling could not have worked out any other way. Viscosity is Pa s (stress times time), velocity m/s (length over time), so stress=Pa s m/(s m)=Pa as it should be. ¹

Note I: Whenever you work out, or type up, a new equation, it is always a good idea to check if the units on both sides make sense.

Note II: The scaling of velocities and stress for a buoyancy driven problem, such as the Stokes sinker discussed below, is entirely different!

1.2 Non-dimensionalization

A complementary approach that also takes into account the order of magnitude of variables is to simplify the governing equations by defining “characteristic” quantities and then dividing all properties by those to make them “non-dimensional”. This way, the non-dimensional quantities that enter the equation on their own should all be of order unity so that the resulting collection of parameters in some part of the equation measures their relative importance.

A classic example for this is based on the Navier Stokes equation for an incompressible, Newtonian fluid. When body forces driving flow are due to temperature T fluctuations in (the Earth’s) gravitational field

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p_d + \eta \nabla^2 \mathbf{v} + \rho_0 \alpha T \mathbf{g} \quad (5)$$

where D is the total, Lagrangian derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \quad (6)$$

\mathbf{v} velocity, ∇ the Nabla derivative operator $\nabla = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$, t time, p_d the dynamic pressure (without the hydrostatic part), η the viscosity, ρ_0 reference density, α , and \mathbf{g} gravitational acceleration. One can now choose (as mentioned before for the Lorenz equations) typical quantities that can be derived from the given parameters such as a ΔT temperature difference, a fluid box height d , and some choice for the timescale. All other characteristic values for physical properties can then be derived from those choices (see, *e.g.*, discussion in [Ricard, 2007](#)).

¹We will always use SI units unless it’s inconvenient for Earth applications, where we might use multiples of SI units such as cm/yr instead of m/s for velocities. Also note that one year has roughly $\pi \cdot 10^7$ s (accurate up to 0.5%), *i.e.* 1 cm/yr is $\approx 3.2 \cdot 10^{-10}$ m/s, and that you should account for leap years for geological time conversions, meaning that $365.25 \times 24 \times 60 \times 60$ is the right number of seconds per year.

Let us assume that we are dealing with a box heated from below and cooled from above, *i.e.* held at constant temperature difference of $\Delta T = T_{top} - T_{bot}$, with no internal heating (Rayleigh-Benard problem). A typical choice for a characteristic timescale is to use the diffusion time that can be constructed from the thermal diffusivity, κ , in the energy equation

$$\frac{DT}{Dt} = \kappa \nabla^2 T \quad (7)$$

(no heat sources) that couples with the momentum equation, eq. (5). Because κ has units of length²/time, any diffusion-related time scale t_d for a given length l has to work out like

$$t_d \propto \frac{d^2}{\kappa}, \quad (8)$$

by dimensional analysis. This relationship is hugely important for all diffusional processes.

Using the characteristic quantities f_c which result from this scaling and using $l = d$, for all variables in eq. (5) and eq. (7), all other properties can be derived, *e.g.*

$$v_c = \frac{d}{t_c} \quad \dot{\epsilon}_c = \frac{v_c}{d} \quad \tau_c = \eta \dot{\epsilon}_c \quad T_c = \Delta T \quad (9)$$

we divide all variables (spatial and temporal derivatives are dealt with like space and time variables) to make them unit-less, non-dimensional $f' = f/f_c$, and eq. (5) can then be written as

$$\frac{1}{Pr} \frac{D'v'}{D't'} = -\nabla' p' + (\nabla')^2 v' - Ra T' \underline{e}_z \quad (10)$$

where we have used $\underline{g} = -g \underline{e}_z$. *I.e.*, all material parameters have been collected in two unit-less numbers after non-dimensionalization, the Prandtl number,

$$Pr = \frac{\eta}{\rho \kappa} = \frac{\nu}{\kappa} \quad (11)$$

and the Rayleigh number ²

$$Ra = \frac{\rho_0 g \alpha \Delta T d^3}{\kappa \eta}. \quad (13)$$

²In the derivation above, we have assumed that the system is heated from below and viscosity is constant. The Rayleigh number is therefore valid for this bottom-heated case only. In Earth's mantle, internal heating (due to decay of radioactive elements) is at least equally important (*e.g.* [Jaupart et al., 2007](#); [Lay et al., 2008](#)). For the case of pure internal heating, the Rayleigh-number is given by

$$Ra_{H'} = \frac{\rho_0 g \alpha H' d^5}{k \kappa \eta}, \quad (12)$$

where k is conductivity and H' is the rate of internal heat generation *per volume* ($H' = \rho_0 H$ where H is per unit mass). We can identify ΔT in (13) with $H' d^2 / k$ which makes sense, since the total heat flux, Q , should scale as $H' d^3$ and $Q \propto k \frac{\Delta T}{d}$. Also, rock viscosity depends on a range of quantities, including temperature and strain-rate, making it imperative to properly (log) average viscosity when computing effective Rayleigh numbers (*e.g.* [Christensen, 1984](#)).

The second way of expressing Pr uses the kinematic viscosity, $\nu = \eta/\rho$, which, like κ , has units of m^2/s ; this makes it clear that Pr measures diffusion of momentum *vs.* diffusion of heat. Ra measures the vigor of convection by balancing buoyancy forces associated with advection against diffusion and viscous drag.

Exercise: Verify this recasting of the Navier-Stokes equation by plugging in the non-dimensionalized quantities.

Often, we then just drop the primes and write the equation like so

$$\frac{1}{Pr} \frac{D\underline{v}}{Dt} = -\nabla p + \nabla^2 \underline{v} - Ra T \underline{e}_z \quad (14)$$

where it is implied that all quantities are used non-dimensionalized (also see sec. ??). This equation may still be hard to solve, but at least we now have sorted all material parameters into two numbers, Ra and Pr .

Note I: The non-dimensional versions of the equations are also the best choice if you want to write a computer program for a physical problem. Using non-dimensionalized equations, all terms should be roughly of order unity, and the computer will not have to multiply terms that are very large in SI units (*e.g.* η) with those that are very small, reducing round-off error (*e.g.* \underline{v} , what is the order of magnitude of η and of $|\underline{v}|$ for mantle convection?).

Note II: This also means that when some geophysicist's convection code spits out, say, velocities, you will have to check what units those have, and more often than not you'll have to multiply by the v_c from above to get back m/s , which you'll then convert to cm/yr .

Note III: You'll also note that a few geodynamics papers will not provide the scaled quantities used so that you can go back to SI units; sometimes this is because the values used for the parameters in the models stray significantly from typical Earth values.

Particularly the Rayleigh number is key for mantle convection, because we typically use the infinite Prandtl number approximation, ($Pr \rightarrow \infty$, why?). In this case, eq. (5) simplifies to the Stokes equation,

$$\eta \nabla^2 \underline{v} = \underline{\nabla} p_d - \rho_0 \alpha T \underline{g}. \quad (15)$$

Both Pr and Ra are discussed below. Fluid dynamics is full of these non-dimensional numbers which are usually named after some famous person because they are so powerful. Any fluid that has the same Ra and Pr number as another fluid will behave exactly the same way in terms of the overall style of dynamics, such as the resulting average temperature structure and up and downwelling morphology.

The actual time scales of convection, *e.g.*, may, however, be very different for two systems at the same Rayleigh number (because of v_c being different). This behavior allows, for example, to conduct analog, laboratory experiments of mantle convection (*e.g.* [Jacoby and Schmeling, 1981](#); [Faccenna et al., 1999](#)). When conducting such experiments, care needs to be taken that all relevant non-dimensional numbers agree between the real Earth problem and the laboratory experiment (*e.g.* [Weijermars and Schmeling, 1986](#)). Also,

when changing length scales and material, different physical effects such as surface tension may matter in the lab, while they are irrelevant for mantle convection in general (see, *e.g.*, sec. 6.7 of [Ricard, 2007](#), for a discussion of Mahagoni convection).

From an analytical point of view, if the non-dimensional quantities are either very large or very small, we can simplify the full equations to more tractable special cases. For a nice and more comprehensive treatment of this section, you may want to refer to [Ricard \(2007\)](#).

1.3 Problems

- a) For all of the following non-dimensional numbers, discuss briefly (2-3 sentences) the processes which these numbers measure, *e.g.* by contrasting system behavior for $Th = 0$ and $Th = \infty$, where Th is some non-dimensional number.

For each number, give numerical estimates for the Earth, at the present day. Document your choices (*i.e.* providing references) for individual parameters before computing joint quantities, mention where you got the estimates from, and what the implications for Earth in terms of the dynamics are. A neat way to organize this might be to use a table for each dimensionless number with appropriate headings (*e.g.* parameter, estimate, reference).

You might have to look up definitions and other reference material, *e.g.* in a geodynamics text, or on Google (*note*: don't trust everything on the web ...). There are no unique answers for this part of the problem set, and you will often have to decide on an example problem for which you'll pick a characteristic quantity such as length. Some answers are actively debated in the literature.

- (i) Rayleigh number for whole and upper mantle convection.
(ii) Peclet number for ridges, slabs, and general mantle convection. The Peclet number is defined as

$$Pe = \frac{dv}{\kappa} \quad (16)$$

with characteristic length d , velocity v , and thermal diffusivity κ .

- (iii) Prandtl number for the mantle and the atmosphere. Once you've figured out the meaning of the Prandtl number, think of the different response of the mantle to an applied pulse of change in plate motion, compared to an applied pulse of heating.
(iv) Reynolds number for the mantle, the ocean, and a tornado. The Reynolds number is defined as

$$Re = \frac{vd}{\nu} = \frac{vd\rho}{\eta}. \quad (17)$$

Note: Take care to distinguish between velocity v , kinematic viscosity $\nu = \eta/\rho$ and dynamic viscosity η .

- (v) Deborah number for the subducting oceanic lithosphere, and for a laboratory experiment on rock deformation. The Deborah number can be defined as

$$De = \frac{t_r}{t_p} \quad (18)$$

where you can use a Maxwell time

$$t_M = \frac{\eta}{\mu} \quad (19)$$

for the relaxation time t_r , and t_p is the time scale of observation. The Maxwell time measures the visco-elastic relaxation time of a body with viscosity η and shear modulus μ (think post-glacial rebound).

- What are characteristic Maxwell times for the crust? The upper mantle?
- b) (i) Consider a solid, sinking sphere of radius a in a fluid of viscosity η and gravitational pull g , and a density Stokes velocity contrast between sphere and fluid of $\Delta\rho$. Solve for the approximate sinking velocity of this “Stokes” sinker by equating the gravitational pull force $F_P = \Delta M g = V \Delta \rho g$ with the shear force acting on the sphere’s area A , $F_S \propto \tau A \propto A \eta \dot{\epsilon}_c$. Here, I’ve used ΔM for the mass anomaly, and V for the volume of the sphere. All equations follow from $F = ma$ and stress = force / area and some geometry.

Note: The full equation for a Stokes sinker is only very weakly dependent on the viscosity of the sinker, η_s , itself, but scales mainly with the ambient viscosity η . For $\eta_s/\eta \rightarrow \infty$ and $\eta_s/\eta \rightarrow 0$, the prefactor changes from 2/9 to 1/3, respectively (see further discussion in *Becker and Faccenna, 2009*, for the subduction context).

- (ii) For flow induced by a Stokes sinker, does the stress scale with η and/or $\Delta\rho$? How does that compare with the velocities?

Note: The scaling of v and τ with combinations of $\Delta\rho$ and η are among the most fundamental relations of mantle dynamics (velocities v might be the plate velocities, dynamic topography scales with τ , for example).

- (iii) Estimate the Stokes velocity by dimensional analysis as in (a), but now assuming that the viscosity of the fluid obeys a power-law,

$$\tau^n \propto \eta \dot{\epsilon} \quad (20)$$

(for rocks, $n \sim 3$) instead of

$$\tau \propto \eta \dot{\epsilon} \quad (21)$$

for Newtonian creep as assumed above. (These equations are written sloppily and don’t have the right units. For correct units, consider a relationship like $\tau (\tau/\mu)^{n-1} = \eta \dot{\epsilon}$, where η is a material parameter, but you may use eq. (20) for the scaling analysis.)

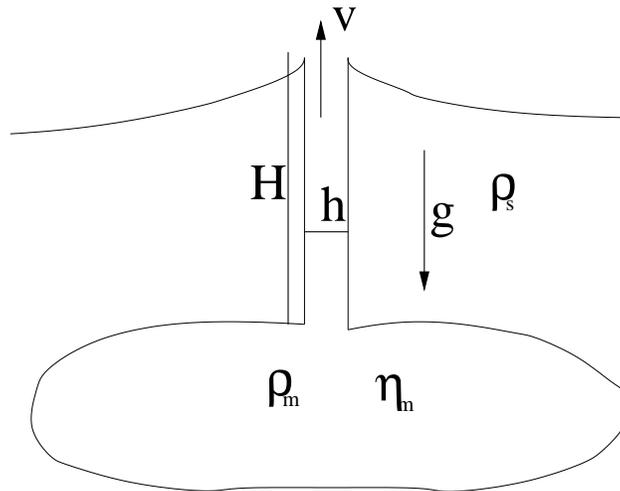


Figure 1: Illustration of the geometry of the volcanic eruption problem.

- (iv) Estimate the rise velocity of a plume head large enough to cause the Deccan traps.
- c) You are moving the top of a fluid layer of height d at constant speed $v(z = d) = v_0$, and the fluid is held fixed at the bottom at $z = 0$. In this case, the laminar solution for the flow velocity is a linear decrease of velocity with depth to $v(0) = 0$ at the bottom.
 - (i) What material parameters set the stress in the fluid? What determines the strain-rate and how does it vary with depth?
 - (ii) Now consider two fluid layers, with the top fluid viscosity larger than the bottom one by a factor of two. Sketch the solution for the dependence of $v(z)$.
- d) Using dimensional analysis, such as used above for the Stokes sinker, estimate the velocity of a volcanic eruption (see Figure below for parameters).
Hint: You might want to proceed by first using the equations for laminar, pressure-driven (look up “Hagen-Poiseuille”) flow in a pipe of radius R , and then estimate the pressure difference from Figure 1.

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