

Figure 1: Stress solution for a sheared, elastic box with an inclusion of different strength (see problem set for details).

1 Exercise: Linear elastic, compressible finite element problem

Reading

- *Hughes (2000)*, secs. 2.7, 2.9 - 2.11, 3.10
- *Dabrowski et al. (2008)*

This FE exercise is again based on the MILAMIN package by *Dabrowski et al. (2008)*. Their “mechanical” solver (incompressible Stokes fluid, to be discussed in the next section) was rewritten for the elastic problem, and simplified to reduce the dependency on packages external to MATLAB.

A highly optimized version of the code that, for example, uses matrix reordering for $\underline{\underline{K}}$ is available from us (this one is closer to the original *Dabrowski et al. (2008)* code). When in-

specting the source codes, you should find many similarities (same mesher, same variable structure, etc.) with last section's 2-D heat equation exercise.

1.1 Implementation of static 2-D elasticity

1.1.1 Problem in strong form

The strong form of the PDE that governs force balance in a medium is given by

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{\underline{f}} = \underline{\underline{0}}, \quad (1)$$

where $\underline{\underline{\sigma}} = \sigma_{ij}$ is the stress tensor and $\underline{\underline{f}}$ a body force (such as due to gravity). (Note that this equation is a general force balance equation in the absence of inertia. You can use it for static elastic deformation, as we do here, or the Stokes fluid flow problem, as we will discuss subsequently. The difference arises in the constitutive law.)

Written in component form as PDEs for the finite element domain Ω for each of the three spatial coordinates i this is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{on } \Omega \quad (2)$$

with essential boundary conditions for displacements $\underline{\underline{u}} = \underline{\underline{g}}$ on Γ_g . Natural boundary conditions for tractions $\underline{\underline{h}} = \underline{\underline{\sigma}} \cdot \underline{\underline{n}}$ shall apply on Γ_h with vector $\underline{\underline{n}}$ normal to the boundary such that

$$u_i = g_i \quad \text{on } \Gamma_{g_i} \quad (3)$$

$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_{h_i}. \quad (4)$$

Here, Γ_h and Γ_{h_i} , and similar for g , denotes that different components of the traction vector may be specified on different parts of the domain boundary Γ .

In the case of linear, elastic behavior, the constitutive law linking dynamic with kinematic properties is given by the generalized Hooke's law

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{C}}} \underline{\underline{\underline{\epsilon}}} \quad \text{or} \quad \sigma_{ij} = C_{ijkl} \epsilon_{kl}, \quad (5)$$

with the elasticity tensor $\underline{\underline{\underline{C}}}$, and the strain-tensor $\underline{\underline{\underline{\epsilon}}}$, computed as

$$\epsilon_{ij} = u_{(i,j)} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (6)$$

Note 1: Notice the definition of the (i,j) derivative short-hand, e.g. operating on $\underline{\underline{u}}$ to get the tensor $\underline{\underline{\underline{\epsilon}}}$, like $u_{(i,j)}$.

Note 2: $\underline{\underline{\underline{C}}}$ is a 4th order tensor and somewhat cumbersome to deal with. Noticing that there are only 6 independent components in $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$, we can write the 21 independent components of $\underline{\underline{\underline{C}}}$ in the Voigt notation, as a 6×6 matrix, $\underline{\underline{C}}_V$. However, this matrix has different definitions (see, e.g. *Browaeys and Chevrot, 2004*, for a discussion), and is not a tensor anymore. I.e. you can do math with it, such as multiplying $\underline{\underline{C}}_V \cdot \underline{\varepsilon}_6$ to get the stress state, where $\underline{\varepsilon}_6$ is a vector that has the six independent components of $\underline{\underline{\varepsilon}}$, appropriately sorted, but you cannot rotate $\underline{\underline{C}}_V$ anymore. For this, the full 4th order $\underline{\underline{\underline{C}}}$ has to be restored before pre- and post-multiplying with the rotation matrices, $\underline{\underline{R}}$ (see eq. ??).

For an isotropic material, the constitutive law between total stress and strain thankfully simplifies to

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} = \lambda \Delta \delta_{ij} + 2\mu \varepsilon_{ij}, \quad (7)$$

where μ and λ are the shear modulus and Lamè parameter, respectively; the latter specifies how incompressible a body is. This formulation uses the isotropic dilation,

$$\Delta = \varepsilon_{ii} = \sum_{i=1}^3 \varepsilon_{ii}, \quad (8)$$

and the Kronecker δ , $\delta_{ij} = 1$ for $i = j$ and zero for $i \neq j$.

The elastic moduli can also be expressed differently, e.g. we can write

$$\lambda = \mu \frac{2\nu}{1-2\nu} = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{with} \quad E = 2\mu(1+\nu), \quad (9)$$

with the Poisson ratio ν and Young's modulus E . (There are only two independent material parameters for isotropic elasticity.) If a block is fixed at the base and loaded in z -directions without constraints, then ν measures the deformation in the horizontal $\nu = -\varepsilon_{xx}/\varepsilon_{zz}$. E measures the stress exerted for the same experiment if the material is not allowed to give way sideways (free-slip in z direction) by $E = \sigma_{zz}/\varepsilon_{zz}$.

The incompressibility, K , is defined as

$$p = -K\Delta = -K\varepsilon_{ii} \quad (10)$$

where p is pressure with

$$p = -\frac{1}{3} \sum \sigma_{ii} = -\frac{1}{3} \sigma_{ii}, \quad (11)$$

and can be computed from

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}, \quad (12)$$

or

$$\mu = \frac{3K(1-2\nu)}{2(1+\nu)}. \quad (13)$$

Note that $\lambda = \mu$ for $\nu = 1/4$, which is often close to values measured for rocks.

1.1.2 Problem in weak form

It can be shown (e.g. *Hughes, 2000*, p. 77ff) that the equivalent weak form formulation of the elastic equilibrium PDE is given by the following statement: Find the displacements \underline{u} for all virtual displacements \underline{w} such that

$$a(\underline{w}, \underline{u}) = (\underline{w}, \underline{f}) + (\underline{w}, \underline{h})_{\Gamma_h} \quad (14)$$

with

$$a(\underline{w}, \underline{u}) = \int d\Omega w_{(i,j)} C_{ijkl} u_{(k,l)} \quad (15)$$

$$(\underline{w}, \underline{f}) = \int d\Omega w_i f_i \quad (16)$$

$$(\underline{w}, \underline{h})_{\Gamma_h} = \sum_{i=1}^3 \left(\int_{\Gamma_{h_i}} d\Gamma w_i h_i \right). \quad (17)$$

Note that unlike the thermal problem, the solution function we wish to obtain using the finite element method is a vector, \underline{u} , rather than a scalar.

1.1.3 Matrix assembly

In the finite element approximation, we then solve for the nodal displacements \underline{d} which approximate \underline{u} within the elements with shape functions N from

$$\underline{\underline{K}} \underline{d} = \underline{F}. \quad (18)$$

The global $\underline{\underline{K}}$ is assembled from the element level by

$$k_{ab}^e = \int_{\Omega^e} d\Omega \underline{\underline{B}}_a^T \underline{\underline{D}} \underline{\underline{B}}_b \quad (19)$$

where a, b are local node numbers. The elemental force vector at local node a is given by

$$f_i^e = \int_{\Omega^e} d\Omega N_a f_i + \int_{\Gamma_{h_i}^e} d\Gamma N_a h_i - \sum_b k_{ab} g_b. \quad (20)$$

$\underline{\underline{B}}$ connects displacements at the nodal level with strains. For 2-D,

$$\underline{\underline{B}}_a = \begin{pmatrix} \frac{\partial N_a}{\partial x} & 0 \\ 0 & \frac{\partial N_a}{\partial z} \\ \frac{\partial N_a}{\partial z} & \frac{\partial N_a}{\partial x} \end{pmatrix}. \quad (21)$$

We can represent the strain tensor $\underline{\underline{\varepsilon}}$ as a strain vector \underline{e} that can be computed from displacements \underline{u} by a gradient operator $\underline{\underline{L}}$ (see eq. 26 for relation to $\underline{\underline{B}}_a$), like

$$\underline{e} = \underline{\underline{L}} \underline{u} \quad \text{or} \quad e_j = L_{jk} u_k. \quad (22)$$

In 2-D, for example,

$$\underline{e} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_x \\ u_z \end{pmatrix}, \quad (23)$$

where the definition of γ_{xy} simplifies the notation, and it is where the “engineering strain” convention arises. Make sure to distinguish it from $\underline{\varepsilon}$, *i.e.* convert with the factor 2 if needed, since

$$\gamma_{xy} = 2\varepsilon_{xy}. \quad (24)$$

Within each finite element the displacements can be obtained by summation over the shape functions for each node a , N_a , times the nodal displacements,

$$\underline{u} = u_k = N_a d_a = N_a d_a^k \quad (25)$$

where d_a is the displacement at the local node a , and d_a^k is the k -th spatial component of this displacement. Then,

$$\underline{e} = e_j = L_{jk} N_a d_a^k = B_{jka} d_a^k = \underline{\underline{B}}_a d_a \quad (26)$$

defines $\underline{\underline{B}}_a$. If we define a stress vector

$$\underline{s} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{pmatrix} \quad (27)$$

with

$$\tau_{xz} = 2\sigma_{xy} \quad (28)$$

in analogy to γ_{xy} , then the (symmetric) elasticity matrix $\underline{\underline{D}}$ can be used to obtain stresses from displacements like

$$\underline{s} = \underline{\underline{D}} \underline{e} = \underline{\underline{D}} \underline{\underline{B}}_a d_a. \quad (29)$$

The $\underline{\underline{D}}$ matrix is a “condensed” version of $\underline{\underline{\underline{C}}}$ (see above),

$$D_{IJ} = C_{ijkl}, \quad (30)$$

where $I, J = 1, 2, \dots, n_{sd}(n_{sd} + 1)/2$ in n_{sd} dimensions, which exploits symmetries in $\underline{\underline{\underline{C}}}$ such that

$$w_{(i,j)} C_{ijkl} u_{(k,l)} = \underline{e}(w)^T \underline{\underline{D}} \underline{e}(u). \quad (31)$$

Note that $\underline{\underline{D}}$ may or may not be identical to $\underline{\underline{\underline{C}}}$ in the Voigt notation, $\underline{\underline{C}}_V$.

Equation (15) can then be written as

$$a(w, u) = \int d\Omega \underline{e}(w)^T \underline{\underline{D}} \underline{e}(u), \quad (32)$$

where $\underline{\epsilon}(\underline{w})$ indicates applying the gradient operator to the virtual displacements, as opposed to $\underline{\epsilon}(\underline{u})$ as in eq. (22).

In the isotropic, 2-D *plane strain* approximation, $\underline{\underline{D}}$ takes the simple form

$$\underline{\underline{D}} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{pmatrix}, \quad (33)$$

where plane strain means that no deformation is allowed in the y direction, $\epsilon_{yy} = 0$.

For the case of *plane stress*, where deformation is allowed and $\sigma_{yy} = 0$,

$$\underline{\underline{D}} = \begin{pmatrix} \bar{\lambda} + 2\mu & \bar{\lambda} & 0 \\ \lambda & \bar{\lambda} + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}, \quad (34)$$

with

$$\bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}. \quad (35)$$

From eq. (35), it is apparent that plane stress reduces the effective, volumetric stiffness of a body, for $\nu = 1/4$, $\bar{\lambda} = 2/3\lambda$, because out of plane deformation is permitted.

1.1.4 Viscous equivalence

The constitutive law for linear viscous flow with viscosity η , and deviatoric stress σ , $\sigma = 2\eta\dot{\epsilon}$, is analogous to the elastic case with $\sigma = 2\mu\epsilon$, assuming the material is incompressible.

The latter can, in theory, be achieved by letting $\nu \rightarrow 1/2$ for which $K/\mu \rightarrow \infty$ such that the linear FE approach can be used to solve simple fluid problems. In practice, however, special care needs to be taken to allow for the numerical solution of the incompressible elastic, or the Stokes flow case, which we discuss in sec. ??.

1.1.5 Exercises

- Make sure you have the MATLAB subroutines `ip_triangle.m`, `shp_deriv_triangle.m`, `generate_mesh.m`, and the `triangle` binary from last section in your working directory. Both shape functions and the mesher will be reused.
- Download `elastic2d_std.m`, a simple linear elasticity solver, and `calc_el_D.m` which assembles $\underline{\underline{D}}$. Also download the driver routine `elastic2d_test.m`. You will have to fill in the blanks.
- Inspect `elastic2d_std.m`, compare with the notes above for linear elasticity, and the heat solver from sec. ??.

- d) Download and inspect `det2D.m`, `inv2D.m`, and `eig2d.m` (for computing the determinant, inverse, and eigen system of 2×2 matrices, respectively). Writing out these operations specifically slightly improves performance compared to using MATLAB's `inv` and `eig` functions. Also download `arrow.m`, which is a routine to plot vectors from the web, and download and inspect `calc_el_stress.m` and `plot2d_strain_cross.m`, which are used to compute element integration node stresses and plot strain- or stress, crossed-vectors symbols for visualization of the stress tensor in the eigen system coordinates, respectively.
- e) Consider a square, homogeneous elastic body with shear modulus $\mu = 1$, Poisson's ratio $\nu = 1/4$ and size 1×1 in x and z directions.

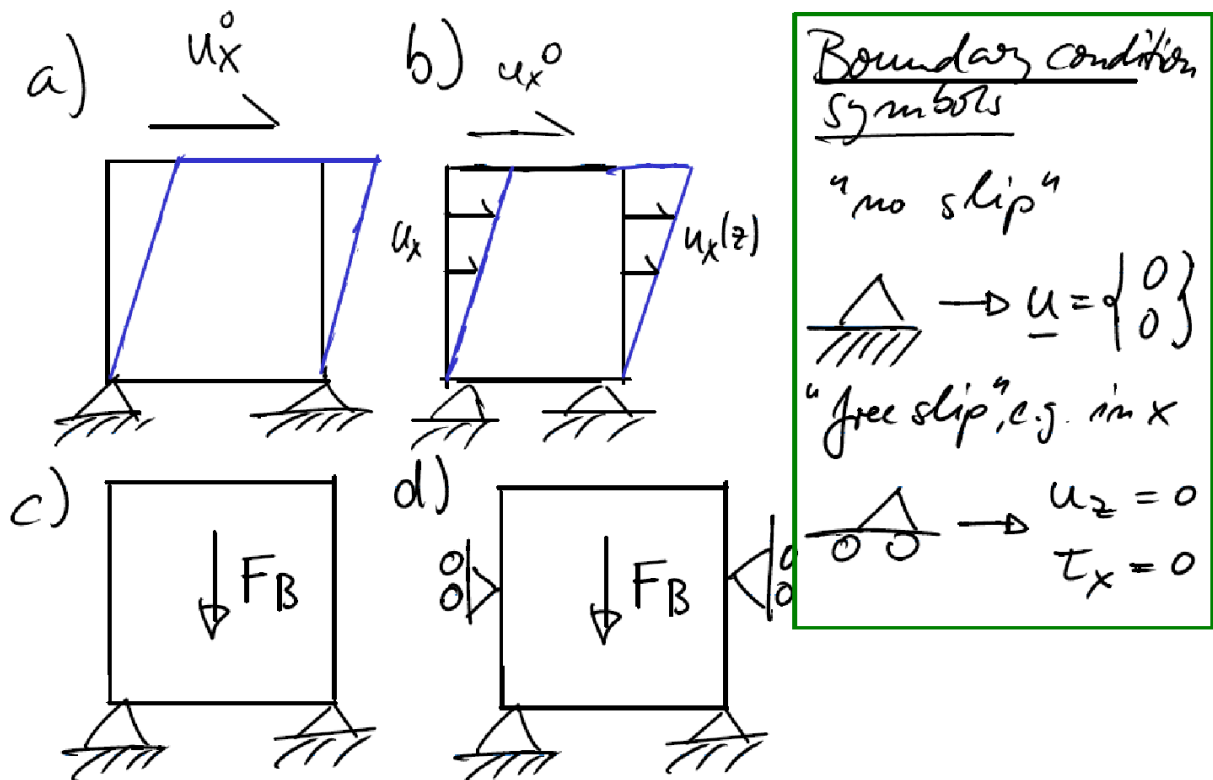


Figure 2: Load case sketches for some of the exercises, along with common symbols for kinematic boundary conditions.

- (i) Assume the body is fixed at the base (zero displacement \underline{u} for all $z = 0$), and sheared with a uniform u_x displacement of $u_0 = 0.1$ at the top ($z = 1$) (Load case a of Figure 2a). Assume the plane strain approximation and zero density (*i.e.* zero body forces). What kind of geologic deformation state does this correspond to? What kinds of displacements would you expect, and how should the major (σ_1) extensional and the major compressional (σ_2) stress axis align?

- (ii) Compute the displacements and stresses using the 2-D FE programs provided. Use linear, three node triangles and experiment with the integration order. Use a coarse mesh with area constraint 0.01 and angle constraint 25° .
For this and each subsequent problem, hand in three plots: 1), of the deformed mesh, indicating the shape of the deformed body, possibly exaggerating the displacements of each node; 2) a plot where the background field (colored) is the amplitude of displacement, and the foreground has displacement vectors, plotted with origin at each original node location; and, 3), a plot of mean (normal) stress (colored in the background), along with extensional and compressional stress axes vector-crosses. The MATLAB routines provided can, with some alterations, perform all of these tasks.
- (iii) Compare the predicted stress and displacements for plane strain and plane stress approximations. Comment.
- (iv) Compare the distorted mesh shape for linear triangles with that for six node, quadratic shape functions. Increase the number of elements and compare the predicted stress fields. Does the displacement and stress field agree with your expectations for this load case?
- (v) Consider Figure 2b and prescribe u_x displacements linearly tapered from $u_x = u_0$ at $z = 1$ down to $u_x = 0$ at $z = 0$. Compare the predicted displacements and stresses with load case Figure 2a. Comment on the stress and displacement fields.
- (vi) Relax the kinematic boundary conditions on the sides and top and include body forces with density $\rho = 1$ at a fixed (no slip) bottom boundary condition (Figure 2c). Compute the displacements and stresses, plot those, and comment.
- (vii) Compute the body force load case of Figure 2d with free-slip (no horizontal displacements, $u_x = 0$, and no “vertical” shear stresses, $\tau_{xz} = 0$) conditions on the left and right sides. Compare the stress field with the previous, unconstrained case and comment.
- f) Consider the square elastic medium in 2-D plane strain plus a centered, spherical inclusion with radius 0.2, shear modulus 0.001. Increase the resolution (*e.g.* use 100 nodes on the outline of the inclusion, 0.001 minimum element area, and 30° triangle edge angle). Load the system as in Figure 2b, compute and plot the stress field, and comment.
- g) *Bonus:* Write a subroutine that computes the stresses at the global node locations, as opposed to the integration points within each element as is currently implemented. Use the nodal stresses and `trisurf` to generate a plot of triangles colored according to their normal stress. Compare with the previous plot.

Bibliography

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