1 Wave propagation

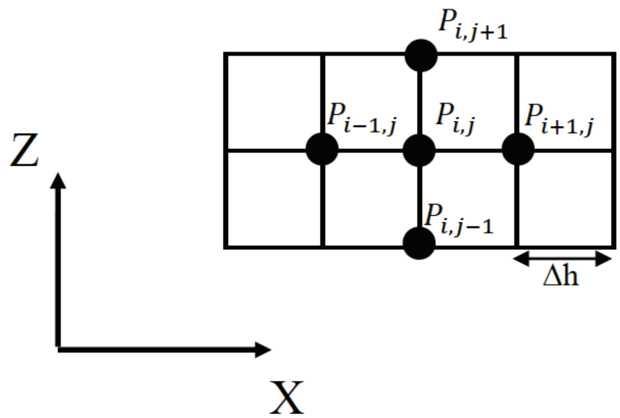

Figure 1: Finite difference discretization of the 2D acoustic problem.

We briefly discuss two examples for solving wave propagation type problems with finite differences, the acoustic and the seismic problem.

1.1 Acoustic problem with standard grid

In an isotropically elastic medium, acoustic wave propagation, where we are only taking care of a single type of wave, can be described by a set of two partial differential equations, leading to a hyperbolic problem. Likewise, we can worry about the propagation of pressure waves in a gas. Newtons 2^{nd} law states that mass × acceleration = force, which for the case of pressure variations in a gas is given by the nagative pressure gradient. Per unit volume, this can be written as

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = \rho \ddot{\underline{u}} = -\underline{\nabla} p. \tag{1}$$

Here, $\underline{u} = \{u_x, u_y, u_z\}$ are the three components of particle displacement, $\dot{}$ and $\ddot{}$ means first and second derivative wrt. to time, respectively, p is pressure, and ρ density. Let us introduce a constitutive law linking pressure to the divergence of displacements,

$$p = -K \underline{\nabla} \cdot \underline{u}, \tag{2}$$

or, taking the second time derivative,

$$\frac{\partial^2 p}{\partial t^2} = -K \underline{\nabla} \cdot \ddot{\underline{u}}, \tag{3}$$

where K is the bulk modulus, or compressibility. If we divide eq. (1) by ρ and take the gradient,

$$\underline{\nabla}\, \ddot{\underline{u}} = \underline{\nabla} \left(\frac{1}{\rho} \underline{\nabla} p \right). \tag{4}$$

Combining the equations, we get

$$\frac{\partial^2 p}{\partial t^2} = K \underline{\nabla} \cdot \left(\frac{1}{\rho} \underline{\nabla} p \right). \tag{5}$$

If we assume that density is constant, we can introduce a parameter, v_B, which turns out to be a velocity of propagation

$$\frac{\partial^2 p}{\partial t^2} = v_B^2 \nabla^2 p, \tag{6}$$

where the bulk sound velocity is

$$v_B = \sqrt{\frac{K}{\rho}}. \tag{7}$$

Simplifying to a 2D case, we have

$$\frac{\partial^2 p}{\partial t^2} = v_B^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} \right). \tag{8}$$

The equation for propogation of SH waves, the transverse components of S waves, in seismology has a similar form as eq. (8):

$$\frac{\partial^2 u}{\partial t^2} = v_{SH}^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{9}$$

where u is the displacement and V_{SH} is the velocity of the SH component.

Likewise, a similar equation also applies for tsunami waves at long wavelengths, in the "shallow water approximation",

$$\frac{\partial^2 \xi}{\partial t^2} = v_{SW}^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial z^2} \right). \tag{10}$$

Here, ξ is the height of the tsunami wave, v_{SW} is the velocity controlled by the water depth H as

$$v_{SW} = \sqrt{gH}. \tag{11}$$

To solve eqs. (8)-(10), with finite differences, we use the mesh shown in Fig. 1. Here, we have $p_{i,j}^n = P(i\Delta h, j\Delta h, n\Delta t)$ and $v_{i,j} = v(i\Delta h, j\Delta h)$ (meaning bulk velocity, v_B). Applying the 2^{nd}-order, second derivative formula to the acoustic wave equation eq. (8),

$$\frac{p_{i,j}^{n-1} - 2p_{i,j}^n + p_{i,j}^{n+1}}{\Delta t^2} = v_{i,j}^2 \left[\frac{p_{i-1,j}^n - 2p_{i,j}^n + p_{i+1,j}^n}{\Delta h^2} + \frac{p_{i,j-1}^n - 2p_{i,j}^n + p_{i,j+1}^n}{\Delta h^2} \right]. \tag{12}$$

After rearranging, we have

$$p_{i,j}^{n+1} = -p_{i,j}^{n-1} + (2 - 4a_{i,j}) p_{i,j}^n + a_{i,j} \left(p_{i-1,j}^n + p_{i+1,j}^n + p_{i,j-1}^n + p_{i,j+1}^n \right), \tag{13}$$

where

$$a_{i,j} = v_{i,j}^2 \frac{\Delta h^2}{\Delta t^2}. \tag{14}$$

Then, the pressure or displacement at time step $n+1$ can be derived explicitly from time step n and $n-1$ as in eq. (13), though two solutions have to be stored. Note that we use 2^{nd}-order second derivatives in eq. (13).

Two considerations are required for choosing suitable time step Δt and spatial step Δh: grid dispersion and stability:

- When waves propagate on a discrete grid, they produces an artificial variation of velocity with frequency, which is called grid dispersion. The higher frequency signals, with slower velocity, are delayed relative to the lower frequency arrivals. This dispersion increases as Δh becomes larger. In other words, a small Δh is required to avoid grid dispersion.

- To achieve an accurate solution, we need at least 12 points per wavelength for space for a scheme with 2^{nd} order accuracy. For a 4^{th} order scheme, a minimum of 6.5 points per wavelength are required. For a fixed frequency, this minimum wavelength is determined by the minimum velocity (v_{min}), so the accuracy of the system is governed by (v_{min}). Following a stability analysis, we can derive the stability requirement here as:

$$\Delta t \leq \frac{1}{\sqrt{2}} \frac{\Delta h}{v_{max}} \tag{15}$$

  where v_{max} is the maximum velocity on the grid.

1.1.1 Exercise 1

a) Program the 2D acoustic wave propagation in standard grid scheme as in Fig. 1 (`wave_acoustic_2D.m`). Study the wavefield and seismograms with different choices of Δt and Δh and demonstrate how Δt and Δh affect the stability and grid dispersion in the program.

b) Introduce heterogeneities in the velocities, such as a thin layer with half velocity, and describe the difference from the isotropic model, especially how this layer affects the observed seismograms. Run the code with the velocity inside the thin layer being zero and explain the result.

1.2 Elastic wave problem with staggered grid

For 2D elastic wave case (P-SV system), force balance and constitutive equations can be written as (*e.g.* Levander, 1988):

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \tag{16}$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}, \tag{17}$$

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z}, \tag{18}$$

$$\tau_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_x}{\partial x}, \tag{19}$$

and

$$\tau_{xz} = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right). \tag{20}$$

Here, (u_x, u_z) are the particle displacements. For seismic waves, they are typically called radial and vertical components, respectively, if they are recorded at surface. Further, τ is the stress tensor, and λ and μ the elastic, Lamé coefficients (μ is shear modulus).

Typically, those equations are solved for particle velocities as $U = \frac{\partial u_x}{\partial t}$ and $V = \frac{\partial u_z}{\partial t}$. Then, the system is transformed into the frst-order hyperbolic system, introducing the abbreviations $\Sigma = \tau_{xx}$, $T = \tau_{zz}$, $\Lambda = \tau_{xz}$,

$$\frac{\partial U}{\partial t} = b \left(\frac{\partial \Sigma}{\partial x} + \frac{\partial \Lambda}{\partial z} \right), \tag{21}$$

$$\frac{\partial V}{\partial t} = b \left(\frac{\partial \Lambda}{\partial x} + \frac{\partial T}{\partial z} \right), \tag{22}$$

$$\frac{\partial \Sigma}{\partial t} = (\lambda + 2\mu)\frac{\partial U}{\partial x} + \lambda \frac{\partial V}{\partial z}, \tag{23}$$

$$\frac{\partial T}{\partial t} = (\lambda + 2\mu)\frac{\partial V}{\partial z} + \lambda \frac{\partial U}{\partial x}, \tag{24}$$

$$\frac{\partial \Lambda}{\partial t} = \mu \left(\frac{\partial U}{\partial z} + \frac{\partial V}{\partial x} \right), \tag{25}$$

with the buoyancy, $b = 1/\rho$.

A typical seismic wave propagation problem needs to deal with medium with variable Poisson's ratio, ν, which can be defined as

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{26}$$

For the special case of $\lambda = \mu$, $\nu = 1/4$, and many rocks have Poisson's ratios not far from $1/4$. For liquids, $\nu \to 0.5$. For seismic wave propagation, this is particularly important when ocean water or the outer core of the Earth are needed to be considered in the problem, which is hard to be resolved with the traditional set up of grid as in Fig. 1.

To satisfy both requirements for stability and grid dispersion at those problems, a *P-SV* staggered-grid scheme is applied. Note the structure of the elastic wave problem, eq. (21)-eq. (25), they allow the stress and particle velocity to be spatially interlaced on the grids as in Fig. 2. The staggered-grid scheme allows the spatial derivative to be computed

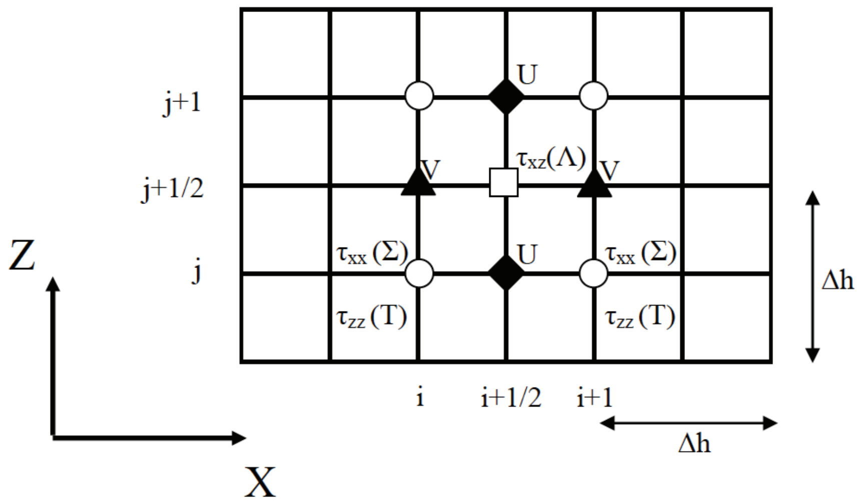

Figure 2: 2D staggered finite difference grid for wave propagation.

to a much higher accuracy (*e.g.* Levander, 1988). This computational aspect is similar to the staggered grid finite difference approach to the Stokes problem, discussed in sec. **??**.

To add the complexity, the stress and velocity field can also staggered in time. We follow the explicit scheme and first update the velocities from time half-steps $k-1/2$ to $k+1/2$, *i.e.* centered on time, $k\Delta t$, using second order, finite difference equations for the first derivatives in eqs. (21)-(25) (Figure 2). Introducing

$$S = \frac{\Delta t}{\Delta h}, \tag{27}$$

we find

$$\begin{aligned} U^{k+1/2}_{i+1/2,j} = U^{k-1/2}_{i+1/2,j} &+ b_{i+1/2,j} S \left(\Sigma^k_{i+1,j} - \Sigma^k_{i,j} \right) \\ &+ b_{i+1/2,j} S \left(\Lambda^k_{i+1/2,j+1/2} - \Lambda^k_{i+1/2,j-1/2} \right), \end{aligned} \tag{28}$$

$$\begin{aligned} V^{k+1/2}_{i,j+1/2} = V^{k-1/2}_{i,j+1/2} &+ b_{i,j+1/2} S \left(\Lambda^k_{i+1/2,j+1/2} - \Lambda^k_{i-1/2,j+1/2} \right) \\ &+ b_{i,j+1/2} S \left(T^k_{i,j+1} - T^k_{i,j} \right). \end{aligned} \tag{29}$$

Then, we advance the stresses from time step k to $k+1$ such that

$$\begin{aligned} \Sigma^{k+1}_{i,j} = \Sigma^k_{i,j} &+ (\lambda + 2\mu)_{i,j} S \left(U^{k+1/2}_{i+1/2,j} - U^{k+1/2}_{i-1/2,j} \right) \\ &+ \lambda_{i,j} S \left(V^{k+1/2}_{i,j+1/2} - V^{k+1/2}_{i,j-1/2} \right), \end{aligned} \tag{30}$$

$$T_{i,j}^{k+1} = T_{i,j}^{k} + (\lambda + 2\mu)_{i,j} S \left(V_{i,j+1/2}^{k+1/2} - V_{i,j-1/2}^{k+1/2} \right) + \lambda_{i,j} S \left(U_{i+1/2,j}^{k+1/2} - V_{i-1/2,j}^{k+1/2} \right), \tag{31}$$

$$\Lambda_{i+1/2,j+1/2}^{k+1} = \Lambda_{i+1/2,j+1/2}^{k} + \mu_{i+1/2,j+1/2} S \left(V_{i+1,j+1/2}^{k+1/2} - V_{i,j+1/2}^{k+1/2} \right) + \mu_{i+1/2,j+1/2} S \left(U_{i+1/2,j+1}^{k+1/2} - U_{i+1/2,j}^{k+1/2} \right). \tag{32}$$

Therefore, to time-evolve the solution for one full Δt, we follow:

a) update velocities from the stress;

b) update the stress from the velocities.

For a homogeneous medium, the stability condition is

$$v_P S = v_P \frac{\Delta t}{\Delta h} < \frac{1}{\sqrt{2}}, \tag{33}$$

where

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{34}$$

is the P-wave velocity. The stability condition is independent of the S-wave velocity

$$v_S = \sqrt{\frac{\mu}{\rho}} \tag{35}$$

because information will propagate at the P wave speed.

To minimize the grid dispersion, the spatial sampling required at least 10 gridpoints per wavelength, which is defined by the v_P, for second order methods such as that of *Virieux* (1986). For a 4^{th}-order approach, the sampling rate can be reduced to 5 grid-points/wavelength (*Levander*, 1988).

Several other issues are also very important for wave propagation in practice:

a) If a boundary condition is not well implemented, the related reflected waves from the boundaries of the domain will affect the results strongly. Depending on the problem, different boundary conditions can be applied to the edges: free-surface conditions, absorbing boundaries (*Clayton and Engquist*, 1977), and the recently widely adopted Perfectly Matched Layer (PML) absorbing boundary (*Collino and Tsogka*, 2001).

b) The source excitation, which initializes the wave propagation, also has to be treated with care. In general, a source can be implemented by simply adding a prescribed source time function to the source mesh. For example, an explosion point source time function $S(t)$ can be added to the 2D elastic case as:

$$\tau_{xx\ or\ zz}(source\ grid) = \tau_{xx\ or\ zz}(FD\ solution\ at\ source\ grid) + S(t)$$

1.2.1 Exercise 2

a) Program the 2D elastic wave propagation in staggered grid scheme as in Fig. 2 (`wave_elastic_staggered_2D.m`). Choose Δt and Δh and describe the wavefield (both vertical and horizontal components) for the model with uniform velocities. Identify the first P and SV arrivals on the recorded seismograms.

b) Include a thin liquid layer ($v_S = 0$) in the model and explain the result. Note for a typical wave propagation problem, the input models are v_P, v_S, and density ρ, so the conversions to λ and μ are required in the program.

1.2.2 Note: Implementation of materal variations

For average densities, $\bar{\rho}$, when considering two materials with ρ_1 and ρ_2, use

$$\bar{\rho} = \frac{\rho_1 + \rho_2}{2}, \tag{36}$$

the arithmetic average. For average elastic properties, *e.g.* shear modulus μ, use

$$\bar{\mu} = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^{-1} \tag{37}$$

(see sec. **??** and, *e.g.*, *Mozco et al.*, 2004, p. 33ff for a discussion of averaging schemes).

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