Deterministic Chaos in two State-variable Friction Sliders and the Effect of Elastic Interactions

Thorsten W. Becker

Department of Earth and Planetary Sciences, Harvard University, Cambridge, Massachusetts, USA

It is demonstrated that a quasistatic slider model with two state-variable rate and state dependent friction shows chaotic dynamics in the deterministic sense. The irregular system behavior can be reduced to a one-dimensional, unimodal mapping which explains the existence of universal period doubling sequences in the stick-slip cycles en route to chaos. This property furthermore allows the approximate prediction of time intervals between sliding events. Lyapunov exponents and periodogram branches depend on the control parameter, the load point coupling, in a self-similar way. Thus, the single slider is a good example of low-dimensional chaos in a homogeneous system, possibly implying a microscopic source of irregularity for earthquakes in nature. However, the sliding patterns of interacting slider models are found to be dominated by perturbation wave phenomena. The wavelength of the asperities that are formed along a slider chain ("fault") increases with the strength of the spring coupling between sliders. This could imply a regularizing effect of interactions, but instabilities prohibit the exploration of the full parameter range for coupled sliders without damping.

1. INTRODUCTION

Laboratory rock sliding experiments initiated by *Di*eterich [1979] and *Ruina* [1980] have led to the establishment of semi-empirical laws that describe the dependence of the friction coefficient on the rate and history of sliding. These "rate and state dependent" laws can explain most of the laboratory observations for a range of materials and parameters [for a recent review, see *Marone*, 1998], and there has been progress in the application to seismicity in nature [e.g., *Dieterich*, 1994; *Scholz*, 1998]. Yet, it is still not entirely clear if and how the microscopic results from sample sizes of $\mathcal{O}(cm)$ and sliding velocities of $\mathcal{O}(\mu m/s)$ can be scaled to real faults with lengths of $\mathcal{O}(km)$ and speeds of $\mathcal{O}(m/s)$ [e.g.,

GeoComplexity and the Physics of Earthquakes Geophysical Monograph 120 Copyright 2000 by the American Geophysical Union Schmittbuhl et al., 1996]. However, I will proceed to examine rate and state friction under the assumption that these laws are also relevant for seismicity in nature. Only simplified quasistatic slider models and elastic interactions will be considered. My models, therefore, do not capture all important processes for faulting in nature, but this simplicity will allow us to gain some physical insight into what might be a part of the complex earthquake system.

In the laboratory experiments, complicated stick-slip sequences have been studied in which slow shear stress increases and vanishing sliding velocities alternate with rapid stress drop and slip. The underlying sliding and history dependence of the friction coefficient, which was formerly reduced to a "static" and a "dynamic" value, can be studied as a microscopic source leading to the observed seismic cycles in fault zones in nature [e.g., *Rice and Tse*, 1986]. Usually, one "state-variable" laws suffice to explain the observed deviations from the traditional concept of static and dynamic friction. The simplest mathematical model which mimics stick-slip with one state-variable friction is a one degree of freedom slider that is spring-coupled to a steadily progressing load point. When inertia is neglected, the resulting quasistatic system of non-linear, ordinary differential equations is two-dimensional (2-D), i.e., it is fully determined by two initial conditions, for example, stress and sliding velocity.

The linear and non-linear stability of this slider system with respect to perturbations has been studied previously [Rice and Ruina, 1983; Gu et al., 1984; Gomberg et al., 1998; Ranjith and Rice, 1999], and I will consider these equations as a starting point for slightly increased complexity. More complicated system behavior can arise when inertia [e.g., Rice and Tse, 1986], continuum interactions [e.g., Horowitz and Ruina, 1989], or a second state-variable [Gu et al., 1984; Gu and Wong, 1994; Zhiren and Chen, 1994] are considered. The latter approach results in the introduction of one additional degree of freedom, and it seems that this is required by some laboratory observations [e.g., Ruina, 1980; Dieterich, 1981; Gu et al., 1984; Blanpied and Tullis, 1986; Gu and Wong, 1994] which show that a single statevariable friction law might be insufficient to interpret the data.

Here I will focus on a quasistatic slider model with two state-variables so that the effect of this constitutive relation can be studied in isolation from other processes such as dissipation or the effect of inertia. Also, I will not be considering pre-existing heterogeneities, be it in the initial conditions (pre-loading) or the material properties. Mathematically, the resulting three dimensional (3-D) phase space dynamics are no longer limited to fixed points (corresponding to stable sliding in the rock friction experiments) or limit cycle oscillations (stick-slip "earthquakes"). It was noted earlier that friction measurements from the laboratory and the corresponding numerical models show period doubling cascades toward irregularity for certain parameter values [Ruina, 1983; Gu et al., 1984; Gu and Wong, 1994]. More recently, Zhiren and Chen [1994] studied a two state-variable slider numerically and suggested that the irregular state which is reached by successive period doubling bifurcations might be chaotic. However, they failed to demonstrate quantitative period doubling conclusively.

This paper will show how the irregular system behavior of a one slider, quasistatic, two state-variable system can be quantified and how certain parameter settings can lead to chaos in the strict deterministic sense. Further, it will be demonstrated that the 3-D dynamics of the chaotic state can be reduced to a one dimensional, unimodal mapping. This allows the approximate prediction of the time intervals between sliding events. As shown by Feigenbaum [1978], the existence of this low-dimensional structure is the reason that a universal period doubling route is realized. The convergence of the control parameter intervals for bifurcations is extracted using "periodograms" that are derived from Poincare sections of the phase space. The resulting plots resemble those for the logistic map [e.g., May, 1976] and demonstrate peculiar features of nonlinear dynamics such as universal period doubling cascades and intermittent period-three windows. Both the period bifurcations and the Lyapunov exponents depend on the controlling load point stiffness in a selfsimilar way. The slider system thus turns out to be a good example of low-dimensional deterministic chaos. However, when next-neighbor interactions for coupled sliders are taken into account, the resulting perturbation waves dominate the sliding heterogeneities, and a regularizing effect is observed with increasing coupling stiffness.

2. THEORETICAL BACKGROUND

This section will discuss the mathematical properties and governing equations of the quasistatic two statevariable slider model and briefly review some measures of irregularity in non-linear dynamical systems.

2.1. Model Definition

We can write a constitutive law for rock interface friction at constant normal stress in terms of the shear stress, τ , as a function of sliding velocity, V, and two "state-variables", θ_1 and θ_2 :

$$\tau(V, \theta_1, \theta_2) = \tau_* + A \ln\left(\frac{V}{V_*}\right) + \sum_{i=1}^2 B_i \ln\left(\frac{V_*\theta_i}{L_i}\right).$$
(1)

The material and environment (e.g., temperature and gouge width) dependent parameters A and B_i control the character of the velocity dependence of friction (weakening or strengthening), and the critical slip lengths, L_i , are relevant for stability. For simplicity, we assume A, B_i , and L_i are constant. Further, V_* and τ_* denote a reference velocity and stress respectively, and the θ_i as well as V depend on time, t. When the friction law is cast in the form of eq. (1), the state-variables can be interpreted as average contact lifetimes for sliding surface roughness. The θ_i were introduced by the experimenters to fit the observed exponential decay processes together with an evolution law like:

$$rac{d heta_i}{dt} = -rac{V heta_i}{L_i}\ln\left(rac{V heta_i}{L_i}
ight),$$
 (2)

where we have used the Ruina ("slip") version [see, e.g., *Marone*, 1998, for discussion]. Inertia-free load point coupling with fixed load point velocity V_0 and coupling stiffness K > 0 can be written as:

$$\frac{d\tau}{dt} = K(V_0 - V),\tag{3}$$

and completes our model equation system for stick-slip (see Figure 1).

We can non-dimensionalize eqs. (1), (2), and (3) by introducing the following variables [slightly modified from *Gu et al.*, 1984]:

$$x = \ln\left(\frac{V}{V_{\star}}\right) \qquad y = \frac{\tau - \tau_{\star}}{A}$$
$$\beta_{i} = \frac{B_{i}}{A} \qquad \kappa = \frac{KL_{1}}{A}$$
$$z = \beta_{2} \ln\left(\frac{V_{\star}\theta_{2}}{L_{2}}\right) \qquad \rho = \frac{L_{1}}{L_{2}}$$
$$T = \frac{V_{\star}}{L_{1}}t.$$

The resulting equations that constitute our model are:

$$\dot{x} = e^x \left((\beta_1 - 1)x + y - z) + \dot{y} - \dot{z} \right)$$
 (4)

$$\dot{y} = (1 - e^x)\kappa \tag{5}$$

$$\dot{z} = -e^{x}\rho\left(\beta_{2}x+z\right), \qquad (6)$$

where dotted quantities are derived with respect to rescaled time T, and we have set the load point velocity V_0 to V_* without loss of generality. Equations (4) to (6) describe the friction system in terms of the nondimensionalized quantities velocity, x, stress, y, and second state-variable, z. In shorthand we can write

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$
 with $\mathbf{x} = (x, y, z)$. (7)

Gu et al. [1984] state that the choice

$$\beta_1 = 1 \quad \beta_2 = 0.84 \quad \text{and} \quad \rho = 0.048$$
 (8)

for three of the four remaining free parameters is appropriate to reproduce experiments of *Ruina* [1980]. Keeping these numbers fixed, the system behavior for a certain initial value is fully characterized by the control



Figure 1. Cartoon of the one degree of freedom slider system. The block is pulled at constant rate V_0 and the friction at the contact interface follows a two state-variable rate and state dependent law.

parameter, the non-dimensionalized stiffness κ .

2.1.1. Stability. The steady state for the model system corresponds to a fixed point in phase space for \mathbf{x} , and the only solution to

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \tag{9}$$

is $\mathbf{x} = \mathbf{0}$ with the physical interpretation of steady sliding along at the speed of the load point. Rice and Ruina [1983] give a general criterion for the linear stability of this fixed point for a class of rate and state type friction systems. Let us consider constant load point velocity, $\mathbf{x} = \mathbf{0}$ (corresponding to slow aseismic creep of a fault), and a sudden perturbation in sliding speed, x, say due to a passing seismic wave. For very stiff coupling with large values of κ , $\mathbf{x} = \mathbf{0}$ is a linearly stable state and small perturbations will die out. Yet, when κ is decreased below a critical value, κ_{cr} , the system undergoes a Hopf bifurcation. This means that the spiraling attraction of the fixed point is transformed into a limit cycle, and further to a repelling spiral where small perturbations from $\mathbf{x} = \mathbf{0}$ move away from the origin without bounds. The system has then become unstable.

Gu et al. [1984] have applied Rice and Ruina [1983]'s criterion to eqs. (4) to (6) and discuss several general properties of the two state-variable friction law. Here, I will only state that their formula for the critical stiffness, $\kappa_{cr}^{(2)}$, translates to my scaling as

$$\kappa_{cr}^{(2)} = \left[\kappa_{cr}^{(1)} + \rho \left(2\beta_{1} + (\beta_{2} - 1) \left(2 + \rho\right)\right)\right] = \left\{4\rho^{2} \left(\kappa_{cr}^{(1)} + \beta_{2}\right) + \left(\kappa_{cr}^{(1)} + \rho^{2}(\beta_{2} - 1)\right)^{2}\right\}^{\frac{1}{2}}\right] / \left[2 + 2\rho\right]. (10)$$

 $\kappa_{cr}^{(1)}$ denotes the critical stiffness of a one state-variable law

$$\boldsymbol{\kappa_{cr}^{(1)}} = \boldsymbol{\beta_1} - 1 \tag{11}$$

and eq. (10) transforms into eq. (11) for $\rho = 1$ and $\beta_2 = 0$. All control parameter values in this paper will

be given in dimensionless form as a fraction of the critical stiffness

$$\kappa' = \frac{\kappa}{\kappa_{cr}^{(2)}},\tag{12}$$

where $\kappa_{cr}^{(2)} \sim 0.08028$ for the choice of parameters (8). Only the normalized quantity κ' is independent of any non-dimensionalization scheme and

$$\kappa' < 1 \tag{13}$$

is the condition for a linearly unstable system.

There has not been a comprehensive parameter study for the two state-variable quasistatic slider equations yet, although *Blanpied and Tullis* [1986] explored stability surfaces and *Gu and Wong* [1994] conducted a range of laboratory and numerical experiments. It is thus not clear if the system properties under consideration here are general, intrinsic features or if they depend strongly on the fine tuning. In addition, other issues such as the quest for the right evolution law (eq. (2)) remain unresolved [e.g., *Marone*, 1998]. At this stage, it seems reasonable to be foremost consistent with the literature [*Gu et al.*, 1984], and I will assume that generic aspects of the friction law (eq. (1)) are captured by the particular parameter choice (8).

2.1.2. Numerical implementation. A step size controlled Cash-Karp Runge-Kutta scheme [*Press et al.*, 1993] was used to solve the system of equations numerically. The integration routine has been benchmarked, was compiled at double precision machine number representation, and set to a precision better than 10^{-8} and an accuracy better than 10^{-7} for single and coupled sliders respectively.

2.2. Measures of the Irregular System State

Various tools have been developed to quantify the irregularity of non-linear dynamical systems such as the one realized by our set of model equations [e.g., Ott, 1993]. Changes in the system variables with time can be described as Lagrangian flow of state points forming a trajectory in phase space from \mathbf{x} to \mathbf{x}' . Calculating Lyapunov exponents for that flow then gives: a) a description of the dynamic stretching of a small sphere of radius r_0 around any initial condition \mathbf{x}_0 , and, by inference, b) a way to determine the dimensionality of an attracting limit object if such a thing exists for bounded trajectories.

Considering a) first, the vigor of mixing and stretching of the initial conditions-sphere tells us about the irregularity of the system evolution from different starting points. The time evolution of the major axes (eigendirections) r_i of the ellipsoid that results from stretching in a 3-D flow can be written in the Floquet form as:

 $r_i(T) \propto r_0 \exp(h_i T)$ with i = 1, 2, 3. (14)

This defines the Lyapunov exponents h_i , so that

$$h_i = \lim_{T \to \infty} \frac{1}{T} \ln\left(\frac{r_i(T)}{r_0}\right). \tag{15}$$

We will sort according to $h_1 > h_2 > h_3$. Under the assumption that the system is ergodic, the time-averaging of eq. (15) should be equivalent to an ensemble average over different initial conditions \mathbf{x}_0 , and the values for h_i are taken as representative of the flow in general. $h_1 > 0$ corresponds to exponentially fast divergence of initial conditions in one direction, the so called "butter-fly effect".

When the system equations are known the h_i can be approximated by averaging the singular values of the Jacobian matrix which gives a linearized version of the flow $\mathbf{F}(\mathbf{x})$. Since the properties of the system lead to rapid growth and shrinkage of matrix elements, a numerical realization of this method has to include frequent re-normalization to obtain accurate results. An alternative approach to quantify stretching has become standard and was proposed by *Benettin et al.* [1980]: for 3-D, the scheme is based on tracking the evolution of three orthogonal vectors, \mathbf{y}_i , which can be approximated by

$$\dot{\mathbf{y}}_{i}(T) = \mathbf{J}|_{\mathbf{x}(T)}\mathbf{y}_{i}(T) \quad \text{with} \quad i = 1, 2, 3,$$
 (16)

where $\mathbf{J}|_{\mathbf{x}(T)}$ is the Jacobian of \mathbf{F} at the position to which initial condition \mathbf{x}_0 has moved at time T. When a Gram-Schmidt ortho-normalization is applied to the \mathbf{y}_i at time intervals δT to avoid overflow, approximations for h_i can be obtained by

$$h_i \approx \frac{1}{l \ \delta T} \sum_{k=1}^l \ln\left(\frac{\alpha_k^{(i)}}{\alpha_k^{(i-1)}}\right). \tag{17}$$

Here, l denotes the number of times the test vectors have been normalized. $\alpha_k^{(i)}$ is the *i*-dimensional "volume" of the parallelepiped spanned by the $\mathbf{y}_1 \dots \mathbf{y}_i$ vectors (i.e., $|\mathbf{y}_1|$, $|\mathbf{y}_1 \times \mathbf{y}_2|$, and $(\mathbf{y}_1 \times \mathbf{y}_2) \cdot \mathbf{y}_3$ for i = 1, i = 2, and i = 3 respectively) before the *k*th normalization took place [see, e.g., *Ott*, 1993, p. 138]. The *Benettin et al.* [1980] method described above was implemented by analytically calculating the Jacobian, evaluating it at the $\mathbf{x}(T)$ location obtained by step size controlled Runge-Kutta and propagating the \mathbf{y}_i by the Euler method. l and δT were usually on the order of 1000 and 50 respectively, and the integration along trajectories was stopped when changes for the h_i converged below 10^{-6} .

The connection between Lyapunov exponents and fractal dimension of objects attracting trajectories is the conjecture of Kaplan and Yorke [1979]. These authors demonstrated that it is likely that the information dimension and the quantity D_{KY} (the Lyapunov or Kaplan-Yorke dimension) are identical. D_{KY} is given by

$$D_{KY} = D + \frac{1}{|h_{D+1}|} \sum_{i=1}^{D} h_i, \qquad (18)$$

where D is the largest integer for which $\sum_{i=1}^{D} h_i > 0$. D_{KY} is therefore a convenient geometrical measure of objects in phase space if Lyapunov exponents can be calculated readily.

3. RESULTS AND DISCUSSION

The next two sections are concerned with the discussion of model results, first from a single slider system and second from interacting spring-coupled slider chains.

3.1. Single Slider Experiments

3.1.1. Period doubling cascades. Previous studies have shown that the two state-variable quasistatic slider system can evolve into a stick-slip limit cycle below the stability bound $\kappa' = 1$ [e.g., Gu et al., 1984]. When the stiffness κ' is continuously decreased, a period-doubling sequence is observed, eventually leading to irregular behavior. Figure 2 shows results from my numerical experiments to illustrate this behavior.

For a stable limit cycle of period two, a slow build-up of stress is followed by a rapid stress-drop in a sliding event, and this is repeated in a strictly periodic fashion. The characteristic zig-zag pattern of stick-slip in the stress versus time plots translates to a deformed limitcycle in phase space, as shown in Figure 2a for $\kappa' = 0.9$. For the parameters given in (8), the system evolves into this state from the unstable fixed point $\mathbf{x} = \mathbf{0}$ when it is slightly perturbed. Large perturbations, on the other hand, lead to growing oscillations and unstable sliding since $\kappa' < 1$. When κ' is decreased further to $\kappa' = 0.86$, the system changes to period four oscillations as in Figure 2b. By looking at the phase space trajectories in Figure 2b, it becomes evident that the folded-loop structure that characterizes period four oscillations could not have been realized in a system with a single state-variable. In that model, the phase space is restricted to 2-D, where uniqueness requires that trajectories do not cross.

Another folding of trajectories forms the period eight cycle when κ' is down to ~ 0.856 (Figure 2c), period sixteen for $\kappa' \sim 0.8552$ (Figure 2d), and so on until an apparently chaotic state is reached at $\kappa' \sim 0.853$ (see, e.g., Figure 5). This period doubling behavior can be quantified using the frequency spectrum of the corresponding time series [Zhiren and Chen, 1994]. In the case of laboratory experiments with incomplete knowledge of the system equations, this is sometimes the only way to proceed in analyzing the system properties [e.g., Libchaber et al., 1982. Detecting the bifurcation values of κ' where new 2^n orbits are formed from the power spectrum can be complicated. For our model I propose the use of simple Poincare sections as a more accurate and straightforward way of quantifying the period doubling cascade.

If we plot the y position of trajectories intersecting the x-z-plane versus κ' (see Figure 2a), the "periodograms" of Figures 3a and 3b arise. Branching lines in these plots correspond to newly created cycles enroute to irregularity on the right hand side. Figure 3 shows all the features which have been discussed for the one-dimensional logistic equation [e.g., May, 1976] such as the period three windows (e.g., at $\kappa' \sim 0.854$) and geometrical self-similarity (compare Figure 3a with 3b). Period doubling bifurcations are now easily detected when the integration of the system equations is run for long enough times (on the order of 25000) to get rid of transients which introduce spurious higher order cycles.

If the two state-variable quasistatic friction system follows the universal period doubling route [Feigenbaum, 1978], the distance factor between the critical values of κ' for the bifurcation sequence n-1, n and n+1,

$$\delta_n = \frac{\kappa'_n - \kappa'_{n-1}}{\kappa'_{n+1} - \kappa'_n},\tag{19}$$

should converge to one of the Feigenbaum numbers:

$$\delta = 4.669201\ldots$$

To test this hypothesis experimentally when κ' is continuously decreased, a bifurcation can be defined as the point where the number of Poincare intersections, 2^n , leaves the plateau of the current cycle of order n. It will then rise to the next level of 2^{n+1} after some numerical transient. Based on this definition, Table 1 was obtained by varying the stepsize in κ' and iteratively narrowing the intervals around the critical values. The δ_n can be observed to converge monotonically to a value ~ 4.48 . I interpret this as satisfactory agreement with the *Feigenbaum* [1978] theory and attribute the small



Figure 2. Phase space trajectories (top) and parts of the corresponding stress, y, versus time, T, series (bottom) for different values of the controlling stiffness κ' . (T scale is offset so that initial transients have decayed.) (a) shows period two, (b) period four, (c) period eight, and (d) period sixteen stick-slip limit cycles. The shaded x-z-plane for part a) illustrates how the Poincare sections for Figures 3a and 3b are obtained; the trajectories were reduced to a projection on z = 0 so that the period two orbit would result in two points at different y values. Trajectories were generated by numerically integrating eqs. (4), (5) and (6) from $\mathbf{x}_0 = (0.05, 0, 0)$ until T = 2000 and plotting the system evolution for the next 1000 timesteps.



Figure 3. (a) Periodogram showing a montage of Poincare sections of the asymptotic system behavior. Obtained by integrating the model equations from $\mathbf{x}_0 = (0.05, 0, 0)$ until T = 20000 and then tracking all Poincare intersections until T = 23000. The apparent distortion of the y scale for the upper tree structure is due to the choice for the Poincare section and could be improved by adjusting the projection further toward the attractor. System behavior for $\kappa' \leq 0.852$ is unstable. (b) Magnification of the box marked by dashed lines in part a).

n	2 ⁿ	κ'	δ_n
1	2	1	
2	4	0.866592	14.09
3	8	0.857127	5.56
4	16	0.855424	4.69
5	32	0.8550609	4.50
6	64	0.8549803	4.48
7	128	0.8549623	

Table 1. Critical numbers for the controlling stiffness κ' at the creation of a 2^n -period limit cycle. The progression factor δ_n is calculated from eq. (19).

deviation to systematic errors, probably due to numerical noise or the way the bifurcations were picked.

My results are at odds with the study of *Zhiren and Chen* [1994] who did not find a clear cut convergence for a similar system of equations. These authors used a different parameter choice, though, and I could not directly reproduce their results, probably because of a misprint in their paper. The discrepancies might therefore be due to actually different system behavior or due to differences in the approach of quantifying the bifurcation locations. Spurious transients might have led to inaccuracies in their Fourier spectrum approach.

By using periodograms, we could therefore show that the quasistatic two state-variable slider follows a universal period doubling road to irregularity, as suggested by *Zhiren and Chen* [1994]. Previously, similar behavior has been found for asymmetrically coupled slider pairs with a simpler friction law [*Huang and Turcotte*, 1990; *Turcotte*, 1997, chap. 11], but this paper makes the first stringent case for a homogeneous friction system.

3.1.2. Unimodal Lorenz mappings. Feigenbaum [1978] demonstrated that quantitative universality in period doubling cascades arises because the dynamics of all qualifying systems can be reduced to a unimodal mapping. It is thus an obvious step to look for the existence of such a mapping in the search for an explanation of the period doubling we found.

As suggested by Lorenz [1963], an irregular time series, in our case the stress y, can be analyzed by plotting the amplitudes of sequential extrema n and n + 1against each other. If the resulting dots of, say, y_{min}^n versus y_{min}^{n+1} trace out a unique graph this indicates that the irregular system has hidden low dimensionality and can be reduced to a 1-D mapping. As a more practical aspect, we could then determine the next minimum value based on the knowledge of the current one, even if a strange looking time-series might suggest otherwise.

Proceeding to construct such a Lorenz-mapping, I plot the y_{min}^n versus y_{min}^{n+1} for the irregular system state at $\kappa' = 0.8525$ in Figure 4a as small dots $(-y_{min})$ is used for convenience). We find that the graph which is traced out is indeed almost unique, i.e. the dots lie basically on a line with small width that does not curve over. Second, the mapping is unimodal; there is only one maximum. The existence of this maximum where the derivative of the mapping goes to zero is in fact the reason for the darker streaks of accumulating points one can find in the irregular region of Figure 3a [e.g., Strogatz, 1994, p. 463]. Third, a fixed point for the 1-D mapping is found at the intersection of the dotted graph and the $y_{min}^{n+1} = y_{min}^n$ line at $y_{min} \sim -1.36$. Since the dot mapping has a slope with absolute value larger than unity at this point, small offsets from it will have grown by the next iteration. This means that the fixed point is unstable and we can expect aperiodic system behavior for all times.

A similar mapping exists for the maximum stress values y_{max} . Since the loading rate is constant, we might then infer that it is possible to predict the "quiet" time intervals ΔT between sliding events (see Figures 2 and 5) on the basis of a simple mapping as well. Figure 4b demonstrates that this is only approximately the case since the finite width of the stress minima and maxima mappings add up to substantial non-uniqueness for the "seismic period" ΔT , especially for $40 < \Delta T < 45$. However, even a rough chance of predicting the timing of the next sliding events might be considered a remarkable property that arises simply from the determinism behind the chaotic time-series of Figure 5.

Summing up, quantitative period doubling with monotonic convergence to $\hat{\delta}$ can be observed as a route to irregularity for the two state-variable friction slider. It was demonstrated that the reason for this universal behavior is the existence of a unimodal mapping between stress extrema. This property allows the approximate reduction of the 3-D dynamics to a 1-D mapping and confirms our suspicion that the system behavior will be aperiodic for all times in the irregular state.

3.1.3. Lyapunov exponents. Figure 5 shows the object that is traced out by trajectories in phase space for the irregular system state at $\kappa' = 0.8525$. It is characteristic for the whole irregular parameter range as indicated in Figure 3 and also found for different values of κ' , say, 0.853. The suggestive interpretation is that it is a strange attractor. With the tools described in section 2.2 we can address the question of classification by calculating the Lyapunov exponents h_i . The numbers obtained for the typical irregular system state



Figure 4. (a) Lorenz mappings of subsequent minima in stress amplitudes, y_{min}^n versus y_{min}^{n+1} , for different values of κ' ($-y_{min}$ is shown for visualization purposes). The single triangle for $\kappa' = 0.9$ corresponds to period two stickslip (compare Figure 2a), all stress minima plot on the fixed point line for $y_{min}^{n+1} = y_{min}^n$. $\kappa' = 0.86$ (diamonds) is a period four limit cycle where y_{min} alternates in amplitude (compare Figure 2b). Further, $\kappa' = 0.856$ (stars) and $\kappa' = 0.8552$ (circles) symbols correspond to period eight and sixteen cycles, respectively (compare Figure 2c and 2d), and the numerous small dots for $\kappa' = 0.8525$ trace out the mapping for the irregular system state (compare Figure 5). Note that the resulting graph is almost unique with some indication of higher dimensionality at $y \sim -1.3$. (b) Mapping of subsequent "quiet" seismic periods, ΔT^m versus ΔT^{n+1} , between steep drops of y for $\kappa' = 0.8525$.

at $\kappa' = 0.8525$ are given in Table 2, where the error range has been estimated on the basis of convergence. The results for the Lyapunov exponents illustrate four points:

First, h_1 is small but positive in the irregular state, indicating exponential stretching of small deviations from a trajectory on the attractor. (We will assume that linear stretching -which could also lead to positive h_1 for finite times- can be ruled out and we ran the simulation long enough to insure this.) This sensitive dependence on initial conditions is one hallmark of deterministic chaos, the others being aperiodicity and determinism. The latter is fulfilled since we specifically set out not to consider any inhomogeneities or random fluctuations. Aperiodicity is indicated by the Lorenz mappings of Figure 4a, which demonstrate that there is no fixed point where the system might get hung up. We can therefore finally classify the irregular system state of the two state-variable slider as chaotic in the strict sense.

Second, $h_1 > 0$, $h_2 \sim 0$ and $h_3 < 0$ holds for chaotic control parameter values. This sign triple appears when trajectories form a strange attractor [e.g., Ott, 1993, p. 136]; then $h_1 > 0$ corresponds to stretching of flow perpendicular to the trajectory along the attractor shape, $h_3 < 0$ arises from the contracting properties of the flow toward the attractor, and the remaining vanishing Lyapunov exponent ($h_2 \sim 0$) corresponds to flow tangential to the trajectory (see also Figure 6).

Third, the sum of all h_i is negative. This means that the flow is on average contractive, corresponding to

$$\sum_{i=1}^{3} h_{i} \approx \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \nabla \cdot \mathbf{F}(\mathbf{x}(T')) dT'$$

< 0. (20)

When we consider a volume V that encompasses all trajectories, we can also write

$$\int_{V} \nabla \cdot \mathbf{F}(\mathbf{x}) dV < 0.$$
 (21)

Eq. (21) could have been derived by integrating the divergence of \mathbf{F} directly, but with the drawback that the resulting expression depends on the integration bounds. Contracting flow all over the phase space implies the existence of a limit state. Since we observe long excursions in phase space nevertheless (see Figure 5) we conclude that this limit state has to be an object with "holes" and can not be a simple geometric surface. We have thus



Figure 5. (a) Strange attractor for the chaotic system state at $\kappa' = 0.8525$ in phase space (side walls show projections). (b) Corresponding stress versus time series. Note how the trajectories almost form a surface resembling a Moebius ribbon in phase space. While the stick-slip sequence is strictly aperiodic and irregular overall, intermittent quasiperiodic sequences can be observed.

Table 2. Lyapunov exponents h_i , sum of all h_i , and Kaplan-Yorke dimension calculated from eq. (18) with D = 2.

Quantity	Value	
$ \frac{\begin{array}{c} h_1\\ h_2\\ h_3\\ \sum_{i=1}^3 h_i\\ D_{KY} \end{array}} $	$\begin{array}{c} 0.01 \pm 10^{-4} \\ 0.0002 \pm 10^{-4} \\ -0.086 \pm 10^{-4} \\ -0.0758 \pm 10^{-4} \\ 2.119 \pm 10^{-3} \end{array}$	

shown that the object that is traced by trajectories in Figure 5 is an attractor.

Fourth, the Kaplan-Yorke dimension indicates that the attractor has a fractal strangeness, meaning a noninteger dimension of ~2.11. Preliminary calculations of the correlation dimension of the attractor, D_c , as defined by Grassberger and Procaccia [1983] indicate that D_c is substantially lower than 2.11, close to ~1.9. A puzzling and as yet unexplained result, since $D_{KY} \approx$ $D_c \gtrsim 2$ is what is usually found for 3-D systems such as the Lorenz attractor [Grassberger and Procaccia, 1983].

After completing this study I became aware of the paper of Niu and Chen [1995], in which the authors calculated Lyapunov exponents in a similar fashion for a two state-variable slider system with different parameter values and a single value of κ' . Niu and Chen's exponents can be rewritten as $h_1 = 0.0124$, $h_2 = 0$ and $h_3 = -0.1094$ in my notation. Hence, the h_i are roughly in agreement and a Lyapunov dimension of 2.11 follows as well. I take this as an indication that the chaotic dynamics might be a stable characteristic of the single slider system for a range of parameters.

3.1.4. The h_i as a function of κ' . This section will be completed by a discussion of the dependence of the Lyapunov exponents on κ' (see Figure 6). By comparing Figure 3 and Figure 6 the values of the h_i can be interpreted in terms of the asymptotic system behavior. For low values of $\kappa' < 0.855$, h_1 (solid line) is positive, indicating a sensitive dependence on initial conditions, hence chaos. The spikes in that parameter range where $h_1 > 0$ correlate with the period three windows of Figure 3 because h_2 (dashed line) is zero in chaotic regions but negative in the periodic windows. For higher values of κ' , h_2 is in general negative but increases to zero repeatedly at the period doubling bifurcations. This κ' dependence is analogous to the first Lyapunov exponent in chaotic 1-D mappings such as the logistic map [e.g., May, 1976]. h_3 (dotted line) mirrors h_1 and h_2 since the sum $\sum h_i$ (dash-dotted line in Figure 6a) is constrained by the contracting property of the flow, and h_3 stays negative for all values of κ' . The overall patterns repeat themselves at different magnification scales (compare Figure 6a and 6b), suggesting that the h_i versus κ' plot shows self-similarity as well.

In summary, the results for the quasistatic single slider model with a two state-variable rate and state dependent friction law have demonstrated that the system is inherently chaotic in the deterministic sense. The model system is universal in that the road to chaos goes through period doubling cycles. If experiments and observations further substantiate the use of two statevariable friction laws to explain fault processes in nature, complex friction laws like the one examined here should be considered a microscopic source for irregular seismicity in the Earth.

3.2. Spring Coupled Sliders

With the results of the last section in mind, one can ask what effect elastic coupling between sliders has on the model seismicity. The inclusion of such interactions is a first step toward accounting for continuum effects in our model, attempting eventually to study the effect of microscopically chaotic friction laws in a homogeneous medium. In the following I will present observations on the resulting stress cycle oscillations.

Figure 7 shows the modified model set-up, a chain of coupled sliders with connecting springs. For mathematical simplicity, only next neighbor interactions are taken into account. This type of interacting slider block-model is similar to other studies in geophysics [e.g., Burridge and Knopoff, 1967; Horowitz and Ruina, 1989; Carlson and Langer, 1989; Espanol, 1994] or in tribology [e.g., Weiss and Elmer, 1996], but it is unique in the choice of friction law. Equation (3) for the *i*-th slider is modified to become

$$\frac{d\tau_i}{dt} = K(V_0 - V_i) + K_{cpl}(V_{i+1} + V_{i-1} - 2V_i). \quad (22)$$

 V_i is the velocity of the slider where the force balance is taken, V_{i+1} and V_{i-1} are the neighboring slider speeds, and K_{cpl} is the spring constant between sliders n-1, n and n+1. For simplicity and symmetry K_{cpl} is assumed constant. K_{cpl} will further be non-dimensionalized in the same way as K and the resulting κ_{cpl} will be expressed as a fraction of κ' so that

$$\kappa_{cpl}' = \frac{\kappa_{cpl}}{\kappa'}.$$
 (23)



Figure 6. (a) Lyapunov exponents h_1 , h_2 , and h_3 (solid, dashed, and dotted line respectively) as well as $\sum h_i$ (dash-dotted line) versus control parameter κ' . (b) Magnification of part a), only the h_i are shown for clarity.



Figure 7. Cartoon of a coupled spring slider system mimicking a fault.

Therefore, if κ'_{cpl} is of order unity or larger the system can be expected to behave homogeneously like a single slider, possibly with a modified effective stiffness. The other case, $\kappa'_{cpl} \ll 1$, corresponds to isolated sliders with very weak interactions. In the continuum analogy, we can think of κ'_{cpl} as the elasticity modul of the material that ruptures along the slider plane, and κ' is a -possibly different- stiffness that connects the material to a fixed driving mechanism, corresponding to asthenospheric loading of a fault in nature. Two examples of coupled models with varying κ'_{cpl} will be described next.

3.2.1. Modulated stick-slip cycles. Figure 8a shows the stress, y, versus time, T, for 10 coupled slider blocks and three different values of the coupling stiffness κ'_{cnl} . The load point stiffness is $\kappa' = 0.965$, a value at which the single slider would show stable period two stickslip. The initial condition of the coupled models is a uperturbation of one slider at T = 0. As can be seen by comparing different traces in Figure 8a, stick-slip oscillations of varying regularity build up over time for every model, regardless of κ'_{cpl} . It is found, however, that a new type of irregularity was introduced and interaction has led to modulation of individual stick-slip oscillators. For weak coupling, $\kappa'_{cpl} = 0.05$, the changes in amplitude and phase ("beating") are strong and an irregular stress drop pattern results. Some traces show that varying amplitude stick-slip alternates with "quiet" periods during which individual sliders creep along with the loading.

Figure 8b shows the average frequency domain representation for the three different coupling experiments of part a), and the power spectrum of a synthetic sawtooth timeseries for comparison. The plots were obtained by Bartlett-tapering all stress timeseries, taking the Fourier transform of the signal [FFT, e.g., *Press et al.*, 1993, p. 504], and averaging over all participating sliders. As expected, the power spectrum of the sawtooth (stick-slip) timeseries although the sliders have slightly less power in the side bands since the transition to sliding is not as abrupt in the friction models

(compare the solid and the dashed lines in Figure 8b). The main frequency, ν^s , of the high coupling "fault" for $\kappa'_{cpl} = 1$ is $\nu^s \sim 0.0317$. This corresponds to a seismic period of $T \sim 31.5$ for sliding events that span the whole fault. From linear perturbation analysis we know that the circular frequency of the periodic orbit at the Hopf bifurcation for the single slider, ω^0 , is equal to the imaginary part of the complex conjugate pair that is found in the three eigenvalues of the Jacobian of the flow **F** at the fixed point $\mathbf{x} = \mathbf{0}$. For $\kappa' = 1$ and the parameters given in (8), the eigenvalues $\Lambda_{1,2,3}$ of J are $\Lambda_{1,2} \sim \pm 0.2093i$, and $\Lambda_3 \sim -0.088$. Since we are still close to the bifurcation, the system frequency ν^s is therefore only slightly detuned from the corresponding eigenvalue frequency at $\kappa' = 1$, $\nu^0 \sim 0.033$, even with interactions.

Turning to the $\kappa'_{cpl} = 0.5$ slider signal (dashed line) in Figure 8b, we observe that its spectral power is richer in frequencies other than the base frequency of the seismic cycle which dominates the $\kappa'_{cpl} = 1$ spectrum. The beating that is observed in the stress drop amplitudes in Figure 8a for $\kappa'_{cpl} = 0.5$ corresponds to two discernible side band entries at AM-modulation periods of $T \sim 670$ and $T \sim 1430$ (small bumps, offset $\Delta \nu \sim \pm 0.0015$ and $\Delta \nu \sim \pm 0.0007$ from ν^s , respectively). Finally, the stronger irregularities observed in the timeseries of $\kappa_{cpl}' = 0.05$ show up as a broad, irregular band of modulation frequencies (dotted line in Figure 8b), as the absence of any dominating cycle modulation pattern in Figure 8a would have led us to suspect. The range of modulation frequencies that is observed for weak coupling corresponds to a broad distribution of the extent of lateral coherence for sliding events, analogous to a broad distribution of seismic event sizes.

In summary, Figure 8 demonstrates that the increase of κ'_{cpl} leads to greater uniformity in the time series until all sliders are almost perfectly synchronized for $\kappa'_{cpl} = 1.0$. This behavior can be interpreted intuitively along the lines of a typical physics textbook example: two spring-coupled pendula. If the coupling between the individual oscillators is weak, an amplitude modulation arises and kinetic energy is transferred back and forth between the pendula. For high coupling, both pendula will swing at the same amplitude and period, acting almost as one pendulum with a modified eigenfrequency.

3.2.2. Slip deficit asperities. Figure 9 presents an alternative view of the sliding history ("seismicity"), now for a larger system with 100 sliders. The plots are grayshaded representations of the slider slip surplus, Σ , in a



Figure 8. (a) Non-dimensionalized stress y versus time T for 10 coupled sliders and three different coupling stiffnesses, κ'_{cpl} . The individual slider traces were offset in y-direction according to their position in the chain, absolute values of y-oscillations are $\sim \pm 2$. κ' is 0.965 and open ended boundary conditions are applied, that is sliders 1 and 10 had only single sided interactions. (b) Fourier domain representation of the time series of part a). All y-traces were Bartlett-window tapered, Fourier transformed, and then averaged over all sliders. Both parts of the figure show the absolute value of the Fourier coefficients versus frequency, while the large plot is a blow-up of the small upper figure around the dominant stick-slip frequency of $\nu^{o} \sim 0.0317$. Solid, dashed, dotted, and dash-dotted lines are for slider time series with $\kappa'_{cpl} = 1$, $\kappa'_{cpl} = 0.5$, $\kappa'_{cpl} = 0.05$, and a synthetic sawtooth signal with $\nu = 0.0317$ respectively. The synthetic series was sampled at the same time intervals as the slider chain and added for comparison. (Note the log-scale which broadens the steep side bands in the FFT of a sawtooth.)

BECKER 19



Figure 9. Slip surplus Σ for 100 coupled sliders and six different coupling stiffnesses, κ'_{cpl} , while κ' is 0.965. The abscissa indicates the slider location in the chain while the ordinate shows time with a scale of T/10. Center plot shows the gray-scale used, while individual slip surplus peaks are between -2 and ~ 20 . The initial condition is a perturbation in sliding speed at the leftmost slider that can be traced as it spreads laterally as an oscillatory instability.

position versus time plane. The neutral state is shown in medium gray shades, denoting sliders whose slip is equal to the amount that would have accumulated by simple sliding along with the load point, V_0t . While darker shading denotes a slip deficit ($\Sigma < 0$) arising from long "healing" times, bright shading marks slip surplus $(\Sigma > 0)$ in which individual sliders overshoot the average offset. The resulting irregular pattern can be considered as an analog to the stress field formed by asperities along a fault subjected to a seismic cycle in nature. (Here, the term asperities is used in a general sense for patches with varying pre-stress along a fault, not necessarily implying changes in the surface properties.) As for the experiment presented in Figure 8, it is evident that higher κ'_{cpl} values result in a more regular seismicity whereas weakly coupled sliders show more small scale irregularity.

Hence, introducing interactions, not surprisingly, has lead to a modulation of the slip characteristics along the coupled oscillator system. On the other hand, increasing the interaction to the same order as the load point coupling has been shown to have a regularizing effect. Similar results were presented by *Horowitz and Ruina* [1989] and *Espanol* [1994] for different types of fault models. That these modulations show up here as well hints at their common origin as an elasticity-coupled oscillator phenomenon.

It is important to note that it was not possible to reproduce the chaotic behavior of the one slider system discussed in the previous section with the undamped inertia-free multi-slider model of this study. Rather than revealing the same period doubling cascades for decreasing κ' as the single slider, numerical simulations of coupled chains of sliders showed unstable behavior for $\kappa < 0.9$. Fault zones in nature are clearly dissipative because of wear on the sliding interface and the radiation of seismic waves. However, the introduction of a regularizing term that might damp the aforementioned instabilities is beyond the scope of this study.

3.2.3. Spatial heterogeneity as a wave phenomenon. As the varying slope of the propagating perturbation wave front for different κ'_{cpl} in Figure 9 indicates, the group velocity with which perturbations in the stick-slip oscillations travel, u, increases with κ'_{cpl} . Since κ'_{cpl} serves as an analog to an elastic modulus for the slider chain, it can be expected that a wave speed analog depends on κ'_{cpl} . For constant material parameters and unit volume, we expect that the phase velocity should scale as $\sqrt{\kappa/m}$ for an elastic medium with inertia, where κ and m denote a stiffness (modul per unit length) and mass respectively. In our quasistatic analysis, the rate and state dependent friction alone plays a role similar to inertia. For example, the "direct effect" in eq. (1), $A \ln(V/V_*)$, gives instantaneous changes in stress for changes in velocity V, thereby representing resistance to acceleration. Assuming then that this virtual inertia effect is independent of κ'_{cpl} to first order and that there is no dispersion we would expect that $u \propto \kappa'_{cpl}^{0.5}$.

u can be estimated in numerical experiments based on the time it takes the initial perturbation to reach a certain slider starting from the initial condition applied at position one. However, it is not clear how this onset of a perturbation wave front should be determined exactly. Figure 10a shows data for five measurements where the onset has been defined as the time when the absolute slip surplus, $|\Sigma|$, of the middle slider is larger than *c* times the maximum slip surplus, $|\Sigma|_{max}$, that is reached by the slider in the remaining experiment. The data points are plotted in a log-log plot for different *c* values between 0.0001% and 1% together with linear regression lines.

We see that the *u* dependence on κ'_{cpl} can be fit by a power law regardless of the value of c. The exponent of the power law does vary, however, between ~ 0.8 for low values of c and ~ 0.2 for c = 1%. We also observe that the slope of the fitted lines converges to ~ 0.8 toward the smallest values of c. I take this as an indication that u scales as $\sim \kappa_{cnl}^{\prime 08}$ for the highest interaction frequencies, which might be expected to travel fastest. The velocity of the main perturbation wave front (higher values of c mean larger amplitudes of Σ) seems to scale with a smaller exponent and the slope of ~ 0.54 for c = 0.1% is close to the prediction of 0.5 based on the hand-waving argument above. Values of c larger than 1% are probably not meaningful if we are interested in determining the perturbation velocity u. It was also found that the exponent of the $u - \kappa'_{col}$ relation does not depend significantly on the system size (number of sliders) or the type of boundary condition. This is in accordance with the wave speed interpretation of u. However, the results for scaling exponents demonstrate that quantitative statements about u are complicated by nonlinearities and dispersion.

We now turn to the lateral heterogeneities in the slip deficit that form after the initial transients in the models of Figure 9. We can observe that small κ'_{cpl} models show short wavelength asperities along the fault. Strong coupling, on the other hand, goes along with longer wavelength, larger scale heterogeneity. (For periodic boundary conditions (not shown), variations in sliding

BECKER 21



Figure 10. (a) Perturbation velocity u versus coupling stiffness κ'_{cpl} . u was obtained from the inverse of the time at which the middle slider in Figure 9 shows an absolute slip deficit, $|\Sigma|$, of magnitude $\geq c|\Sigma|_{max}$. $|\Sigma|_{max}$ denotes the maximum of $|\Sigma|$ during the remainder of the experiment and five measurements between c = 0.0001% and c = 1% are shown. The data was fit with a power law for each experiment and the exponents are given in the legend together with formal *a posterior* estimates of 1σ -uncertainty using χ^2 . (b) Average spectral power for slip asperities in the spatial domain, $\bar{F}(\Sigma)$, versus spatial frequency, f. $\bar{F}(\Sigma)$ was computed from the models shown in Figure 9 by taking the FFT of the Bartlett-window tapered slip deficit at constant times and averaging over 100 timesteps. Distributions for six different experiments with $\kappa'_{cpl} = 0.05$, 0.1, 0.2, 0.5, 1, 1.5, and $\kappa'_{cpl} = 4$ are shown. Also indicated are the center of mass, P, values as defined in eq. (24) for $\kappa'_{cpl} = 1$ and $\kappa'_{cpl} = 0.5$. (c) Quantitative analysis of the power spectra of part b). Circle symbols denote estimated exponents, a, of a power-law frequency decay $\propto 1/f^a$, obtained by linear regression to get the slope of the spectra in part b) within the range $0.06 \leq f \leq 0.2$. Errorbars are formal uncertainties based on χ^2 (linear scale for a). Square symbols indicate the inverse of the first moment, 1/P (log scale for 1/P). The solid line is a fit for 1/P in the range $0.05 \leq \kappa'_{cpl} \leq 1$, indicating that $1/P \propto \kappa'_{cpl}$.

history are almost entirely suppressed for $\kappa'_{cpl} \gtrsim 1$.) If we assume a constant spatial modulation (or interaction) frequency f_m for lateral cycle perturbations at constant κ'_{cpl} and constant slider number, an average dominant wavelength of the lateral slip irregularity, $\bar{\lambda}$, should scale with u when $u = f_m \bar{\lambda}$. This means that higher velocities will tend to organize the slip pattern over larger length scales. Higher coupling should, therefore, not only bring about a higher perturbation velocity u as was demonstrated in Figure 10a, but also longer wavelength asperities, as observed in the Σ -patterns of Figure 9.

While a spectral analysis of the sliding heterogeneity with time was already presented in section 3.2.1, I will now proceed to analyze the spatial frequency content of the asperities that are shown in Figure 9. Figure 10b is a plot of the average spectral power of the slip deficit asperities, $\bar{F}(\Sigma)$, versus frequency, f $(0 \leq f \leq f_{\text{Nyquist}} = 0.5)$, for seven different models with varying coupling stiffness. The graphs were obtained by averaging the spectral density estimate for the last 100 timesteps in each experiment. Figure 10b confirms that high κ'_{cpl} seismicity is equivalent to a concentration of power in the lower frequencies, and weaker coupling results in an emphasis of the short spatial periods. The $\kappa'_{cnl} = 4$ spectrum corresponds to the synchronous end-member case with almost no variations along strike, similar to the $\kappa'_{col} = 1$ experiment for a ten slider model that was shown in Figure 8a.

The frequency distributions are roughly linear in the log-log-plot for spatial frequencies in the range 0.06 $\leq f \leq 0.2$, and the rate of decay (the negative slope of the power-law part of the distributions) increases with increasing coupling. It is also found that there is a transition from a rough power spectrum at low κ'_{cpl} (e.g., solid line in Figure 10b for $\kappa'_{cpl} = 0.05$) to a smoother distribution for high coupling experiments (e.g., dash-dotted line for $\kappa'_{cpl} = 1$); the character of the spectral power distributions changes at $\kappa'_{cpl} \sim 0.7$. Also, the $\kappa'_{cpl} = 1$ spectrum is somewhat of an exception in that is does not follow the general trend for $\kappa'_{cpl} = 1$ than for $\kappa'_{cpl} = 0.5$.

To quantify the observation that the slope of the power spectra decays as a function of κ'_{cpl} , I fitted a power law $1/f^a$ to the data in the interval $0.06 \le f \le 0.2$. The circle symbols in Figure 10c indicate the values for that best-fit exponent, a, versus κ'_{cpl} . We see that the slope changes from a very slow, 1/f-type decay for

 $\kappa'_{cpl} = 0.05$ with $a \sim 0.5$ to more rapid power cut-off with exponents ~ 6.5 at $\kappa'_{cpl} = 0.7$. For larger κ'_{cpl} , Figure 10c indicates a saturation at large scale organization, corresponding to $\sim 1/f^6$. However, we observe considerable scatter and a decrease in the slope toward synchronous sliders at $\kappa'_{cpl} = 4$.

Finally, the first moment, P, of the $\overline{F}(\Sigma)$ curves,

$$P = \frac{1}{M} \int_0^{0.5} f\bar{F}(\Sigma) df \quad \text{with} \qquad (24)$$

$$M = \int_0^{0.5} \bar{F}(\Sigma) df, \qquad (25)$$

can be calculated as a measure for the "center of mass" of the spectral power distributions (see also Figure 10b). Square symbols in Figure 10c denote the inverse of P so that higher values of 1/P correspond to more power in the lower spatial frequencies. We can observe that 1/Pincreases with κ'_{cpl} , for $\kappa'_{cpl} \leq 1$ roughly as a power law with $\kappa'_{cpl}{}^{0}{}^{6}$. For larger κ'_{cpl} the plot again indicates some scatter and super-power law increase of 1/P when the systems gets close to synchrony. The spectral analvsis has therefore demonstrated that we can interpret the increase in the length scale of slip organization for increasing κ'_{cpl} as a result of longer perturbation wavelengths. 1/P scales roughly in the same way with κ'_{col} as the perturbation velocity u. For $\kappa'_{cpl} \gtrsim 1$ we observed some qualitative differences between high and low κ'_{crl} seismicity which result from the coherent sliding events of the asymptotic end-member state without variations in the seismic cycle.

Summing up, coupling modifies the single slider dynamics. The extreme cases of weak coupling with small wavelength cycle perturbations and very strong coupling where sliders move in synchrony border a range in which irregular sliding histories form as a result of sustained modulation waves.

4. LIMITATIONS AND IMPLICATIONS OF THE MODEL

Investigations of the single quasistatic slider system demonstrate that two state-variable friction laws can lead to deterministic chaos in a homogeneous system. Laboratory rock friction can therefore serve as yet another example of the peculiarities of nonlinear dynamics: while some aspects show the underlying determinism (e.g., map-predictions of the seismic period, Figure 4), any irregularity will be locally amplified by the sensitive dependence on initial conditions. If for nothing else, the system can be viewed as a tutorial for low-dimensional chaos from the Earth sciences, to be compared with other irregular systems as reviewed, for example, by *Turcotte* [1997].

In the following, I will discuss the possible effect of inertia and elaborate on conclusions one can draw from the results of interacting sliders.

4.1. Inertia

The influence of inertia on the dynamics of the slider block system might be important, yet it has been neglected in all the above models. We can estimate that the effect of mass will be twofold. First, inertia will tend to destabilize the system. As discussed by Rice and Ruina [1983], including a non-zero mass in the linear stability analysis of any single slider rate and state friction type system results in a decrease in the critical stiffness. A material with a constant κ' analog modulus will thus be more unstable with inertia than without. However, as calculations of *Rice and Tse* [1986] for single state-variable sliders show, the overall stickslip characteristics of systems with inertia are similar to the quasistatic case, although the stress drop events are modified by the induced dynamic overshoot. Further, Gu and Wong [1994] demonstrated that period doubling cascades are also observed in inertial systems with two state-variables. In their models, the irregular parameter range as in Figure 3 toward lower values of κ' was not terminated by unstable sliding but rather by quasiperiodic system behavior.

Second, mass introduces another degree of freedom in the slider equations. Since three mathematical dimensions are a necessary condition for chaos, one statevariable sliders with inertia would also be possible candidates for a microscopic source of irregularity. I am not aware of any studies that show chaotic behavior for less than two state-variables for single slider rate and state friction systems. Yet, two dynamical sliders with asymmetric coupling were demonstrated to undergo period doubling cascades [Huang and Turcotte, 1990]. Based on the results of the studies mentioned above, we can state that the inclusion of inertia leads to modified system behavior, including shifted stability bounds. However, since inertia alone apparently does not change the overall characteristics, studying the simpler quasistatic system should be a good start, especially given the complexity that is already unraveled at this level of simplification.

As the comparison with the work of Huang and Turcotte [1990] further shows, even for simple slider systems two possible origins of irregularity are found which might lead to the seismicity that is observed in nature: either a complicated friction law working at the microscopic level might be the cause (as in the studies of Gu et al. [1984], Gu and Wong [1994], Zhiren and Chen [1994], and in this work), or irregularity might arise with simpler microscopic laws but heterogeneous interactions. This leads to the discussion of effects that arise when sliders are coupled.

4.2. Interaction

Seismicity in the Earth shows some regular features of almost periodic earthquake recurrence [e.g., Bakun and McEvilly, 1984], many more examples of irregular seismic cycles that are only quasiperiodic (large scale heterogeneity) [e.g., Sieh, 1981], and power-law magnitude-frequency distributions of events. The latter Gutenberg and Richter [1949]-type (GR type) size distributions might indicate criticality in the sense of Bak et al. [1988], with possible consequences for earthquake interaction ranges and predictability [e.g., Sornette and Sornette, 1989].

Assuming that the complex, possibly chaotic laboratory friction laws we studied have relevance for the behavior of fault zones in nature, we can try to evaluate whether a homogeneous fault in which these laws apply still produces regular seismic cycles in a continuum. If it does, then other mechanisms such as spatial heterogeneity of material parameters (noise input), geometrical complexity of fault traces [fractal grounds to start from, e.g., *King*, 1983], or mechanical fault interactions [e.g., *Harris*, 1998, and references therein] might be more important in leading to the observed irregularities in nature.

Previous studies have incorporated simpler friction laws in slider blocks [see, e.g, *Elmer*, 1996; and *Turcotte*, 1997, chap. 17, for reviews], or more sophisticated continuum models [e.g., *Horowitz and Ruina*, 1989; *Rice*, 1993; *Shaw*, 1995; *Cochard and Madariaga*, 1996]. However, the conditions under which fully dynamical models of faults produce periodic seismic cycles, large scale irregularity, or GR-type characteristics are still debated [e.g., *Rice and Ben-Zion*, 1996]. It appears that simple, one state-variable friction laws generically produce larger scale irregularity from a homogeneous model, but GR-statistics seem to be only

24 CHAOS IN FRICTION

the outcome of a small parameter range that might not be realized on Earth [Shaw and Rice, 1999].

The slider models from the previous section are an attempt to contribute to this discussion on a very simplified level. Coupled sliders show slip histories which do not result from the individual friction law's characteristics but are dominated by interaction effects. Similar findings were reported by Horowitz and Ruina [1989], and Espanol [1994] discussed the transition from periodicity to soliton-like solutions as in my Figure 9 for a velocity weakening friction law. More recently, de Sousa Vieira [1996] examined changes in the magnitude-frequency distributions of sliding events as a function of the coupling stiffness for a standard springblock model with inertia and simple friction. In my quasistatic models, the variations in stick-slip cycle amplitude are a coupled oscillator effect. They can be described by a lateral interaction mechanism whose wavelength increases with the strength of coupling. Higher coupling was therefore observed to have a regularizing effect, leading to larger wavelength asperities and more regular slip patterns. The observed mechanism of regularization might have general relevance for seismicity in nature since a variety of models with different friction laws, with and without inertia, appear to show similar features.

My slider experiments have various shortcomings, such as the limitation to a single degree of freedom for motion, discrete blocks rather than a continuum model, and the limited parameter range that could be explored without damping. It is hence not possible to conclusively quantify the extent to which two state-variable friction can act as a source of chaos in coupled systems at this point. Given the small parameter range in which chaos is observed for a single slider and the observed regularizing effect of interactions, my conjecture is that elastic interactions on different scales dominate in the Earth, reducing the effect of microscopic chaos in friction, and leading to larger scale irregularity.

5. CONCLUSION

Two state-variable rate and state dependent friction laws that are derived from laboratory rock sliding experiments were shown to result in deterministic chaotic behavior for a simplified quasistatic model. Since a unimodal mapping can be derived from aperiodic time series, universal period doubling cascades are observed as a route to chaos. The single slider model is thus a good example with which to demonstrate bounds on the predictability of model rupture events. Interaction models imply that the microscopic friction law is not as important in coupled sliders. Other sources of irregularity such as mechanical interaction between faults might be more important in nature.

Acknowledgments. The author thanks the anonymous reviewer for his suggestions as well as Sascha Hilgenfeldt, Richard Kronauer, Richard O'Connell, Svetlana Panasyuk, James Rice, and especially Simone Bethge and James Kellogg for discussion and encouragement. Funding for this study was provided by the German Academic Exchange Service (DAAD) under a "Doktorandenstipendium HSP-III".

REFERENCES

- Bak, P., Tang, C., and Wiesenfeld, K., Self-organized criticality, *Phys. Rev. A*, 38, 364–373, 1988.
- Bakun, W. and McEvilly, T., Recurrence models and Parkfield, California earthquakes, J. Geophys. Res., 89, 3051– 3058, 1984.
- Benettin, G., Galgani, A., Giorgilli, A., and Strelcyn, J.-M., Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems: a method for computing all of them. Part 2: Numerical application, *Meccanica*, 15, 21, 1980.
- Blanpied, M. L. and Tullis, T. E., The stability and behavior of a frictional system with a two state variable constitutive law, *Pure Appl. Geophys.*, 124, 415-444, 1986.
- Burridge, R. and Knopoff, L., Model and theoretical seismicity, Bull. Seism. Soc. Am., 57, 341-371, 1967.
- Carlson, J. M. and Langer, J. S., Properties of earthquakes generated by fault dynamics, *Phys. Rev. Lett.*, 62, 2632– 2635, 1989.
- Cochard, A. and Madariaga, R., Complexity of seismicity due to highly rate-dependent friction, J. Geophys. Res., 101, 25,321-25,336, 1996.
- de Sousa Vieira, M., Exponential distributions in a mechanical model for earthquakes, *Phys. Rev. E*, 54, 5925–5928, 1996.
- Dieterich, J. H., Modeling of rock friction 1. Experimental results and constitutive equations, J. Geophys. Res., 84, 2161–2168, 1979.
- Dieterich, J. H., Mechanical behavior of crustal rocks, in Mechanical behavior of crustal rocks: the Handin volume, vol. 24, Geophysical Monograph, edited by N. L. Carter, M. Friedman, J. M. Logan, and D. W. Stearns, pp. 103– 120, AGU, Washington, D. C., 1981.
- Dieterich, J. H., A constitutive law for rate of earthquake production and its application to earthquake clustering, J. Geophys. Res., 99, 2601-2618, 1994.
- Elmer, F.-J., Is Self-Organized Criticality possible in dry friction?, in *Physics of sliding friction*, edited by B. N. J.

Persson and E. Tosatti, pp. 433–447, Kluwer Academic, Dordrecht, The Netherlands, 1996.

- Espanol, P., Propagative slipping modes in a spring-block model, Phys. Rev. E., 50, 227-235, 1994.
- Feigenbaum, M. J., Quantitative universality for a class of nonlinear transformations, J. Stat. Phys., 19, 25, 1978.
- Gomberg, J., Beeler, N. M., Blanpied, M. L., and Bodin, P., Earthquake triggering by transient and static deformations, J. Geophys. Res., 103, 24,411-24,426, 1998.
- Grassberger, P. and Procaccia, I., Measuring the strangeness of strange attractors, *Physica D*, 9, 189, 1983.
- Gu, J.-C., Rice, J. R., Ruina, A. L., and Tse, S. T., Slip motion and stability of a single degree of freedom elastic system with rate and state dependent friction, J. Mech. Phys. Solids, 32, 167-196, 1984.
- Gu, Y. and Wong, T.-F., Nonlinear dynamics of the transition from stable sliding to cyclic stick-slip in rock, in Nonlinear dynamics and predictability of geophysical phenomena, vol. 83, Geophysical Monograph, edited by W. I. Newman, A. Gabrielov, and D. L. Turcotte, pp. 15–35, AGU, Washington, D. C., 1994.
- Gutenberg, B. and Richter, C., Seismicity of the Earth and Associated Phenomena, Princeton Univ. Press, Princeton, 1949.
- Harris, R. A., Introduction to special section: Stress triggers, stress shadows, and implications for seismic hazard, J. Geophys. Res., 103, 24,347-24,358, 1998.
- Horowitz, F. G. and Ruina, A. L., Slip patterns in a spatially homogeneous fault model, J. Geophys. Res., 94, 10,279– 10,298, 1989.
- Huang, J. and Turcotte, D. L., Are earthquakes an example of deterministic chaos?, *Geophys. Res. Lett.*, 17, 223-226, 1990.
- Kaplan, J. L. and Yorke, J. A., Chaotic behavior of multidimensional difference equations, in *Functional Differential Equations and Approximations of Fixed Points*, vol. 730, *Lecture Notes in Mathematics*, edited by H.-O. Peitgen and H.-O. Walter, pp. 204–227, Springer, Berlin, 1979.
- King, G. C. P., The accomodation of large strains in the upper lithosphere of the earth and other solids by self-similar fault systems: the geometric origin of b-value, *Pure Appl. Geophys.*, 121, 761–815, 1983.
- Libchaber, A., Laroche, C., and Fauve, S., Period doubling cascade in mercury, a quantitative measurement, J. Physique. Lett., 43, L211–L216, 1982.
- Lorenz, E. N., Deterministic nonperiodic flow, J. Atmos. Sci., 20, 130, 1963.
- Marone, C., Laboratory-derived friction laws and their application to seismic faulting, Annu. Rev. Earth Planet. Sci., 26, 643-696, 1998.
- May, R. M., Simple mathematical models with very complicated dynamics, *Nature*, 261, 459, 1976.
- Niu, Z.-R. and Chen, D.-M., Lyapunov exponent and dimension of the strange attractor of elastic frictional system, *Acta Seimol. Sinica*, 8, 575–584, 1995.
- Ott, E., Chaos in dynamical systems, Cambridge Univ. Press, Cambridge, UK, 1993.
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flan-

nery, B. P., Numerical Recipes in C: The Art of Scientific Computing, Cambridge Univ. Press, Cambridge, UK, 2nd ed., 1993.

- Ranjith, K. and Rice, J. R., Stability of quasi-static slip in a single degree of freedom elastic system with rate and state dependent friction, J. Mech. Phys. Solids, 47, 1207–1218, 1999.
- Rice, J. R., Spatio-temporal complexity of slip on a fault, J. Geophys. Res., 98, 9885–9907, 1993.
- Rice, J. R. and Ben-Zion, Y., Slip complexity in earthquake fault models, Proc. Natl. Acad. Sci. USA, 93, 3811–3818, 1996.
- Rice, J. R. and Ruina, A. L., Stability of steady frictional slipping, J. Appl. Mech., 50, 343-349, 1983.
- Rice, J. R. and Tse, S. T., Dynamic motion of a single degree of freedom system following a rate and state dependent friction law, J. Geophys. Res., 91, 521-530, 1986.
- Ruina, A. L., Slip instability and state variable friction laws, J. Geophys. Res., 88, 10,359–10,370, 1983.
- Ruina, A. L., Friction Laws and Instabilities: A Quasistatic Analysis of some Dry Frictional Behavior, Ph. D. thesis, Brown University, Providence, 1980.
- Schmittbuhl, J., Vilotte, J.-P., and Roux, S., Velocity weakening friction: A renormalization approach, J. Geophys. Res., 101, 13,911–13,917, 1996.
- Scholz, C. H., Earthquakes and friction laws, Nature, 391, 37–42, 1998.
- Shaw, B. E., Frictional weakening and slip complexity in earthquake faults, J. Geophys. Res., 100, 18,239–18,251, 1995.
- Shaw, B. E. and Rice, J. R., Existence of continuum complexity in the elastodynamics of repeated fault ruptures (abstract), EOS, Trans. AGU, 80(46), Fall Meet. Suppl., F688, 1999.
- Sieh, K., A review of geological evidence for recurrence times of large earthquakes, in *Earthquake Prediction, an International Review*, vol. 4, *M. Ewing series*, edited by D. Simpson and P. Richards, pp. 209–216, AGU, Washington D. C., 1981.
- Sornette, A. and Sornette, D., Self-organized criticality and earthquakes, *Europhys. Lett.*, 9, 197–202, 1989.
- Strogatz, S. H., Nonlinear Dynamics and Chaos, Addison-Wesley, Reading, MA, 1994.
- Turcotte, D. L., Fractals and chaos in geology and geophysics, Cambridge Univ. Press, Cambridge, UK, 2nd ed., 1997.
- Weiss, M. and Elmer, F.-J., A simple model for wearless friction: the Frenkel-Kontorova-Tomlinson model, in *Physics of sliding friction*, edited by B. N. J. Persson and E. Tosatti, pp. 163–178, Kluwer Academic, Dordrecht, The Netherlands, 1996.
- Zhiren, N. and Chen, D., Period-doubling bifurcation and chaotic phenomena in a single degree of freedom elastic system with a two-state variable friction law, in Nonlinear dynamics and predictability of geophysical phenomena, vol. 83, Geophysical Monograph, edited by W. I. Newman, A. Gabrielov, and D. L. Turcotte, pp. 75–80, AGU, Washington, D. C., 1994.

26 CHAOS IN FRICTION

Thorsten W. Becker, Department of Earth and Planetary Sciences, Harvard University, 20 Oxford St., Cambridge, MA 02138, USA. (e-mail: becker@eps.harvard.edu)

Note added in proof. In a work of which I was unaware until after the preparation of the final copy of this article, *Shkoller and Minster* [Shkoller, S. and Minster, J.-B., Reduction of Dietrich-Ruina attractors to unimodal maps, *Nonlin. Process. Geophys.*, 4, 63–69, 1997] constructed unimodal mappings directly from the attractor of the two state-variable system. This alternative to my time-series approach yields comparable results, confirming that chaos can be an intrinsic feature in dry friction.