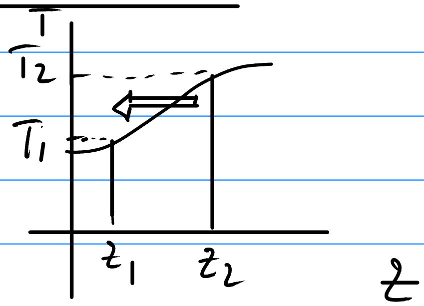
S34

Heat transport

- radiation
- advection
- conduction

} combine to get convection

Conduction



heat is transported along gradients of temperature

$$q \propto \frac{T_2 - T_1}{z_2 - z_1} = \frac{\Delta T}{\Delta z}$$

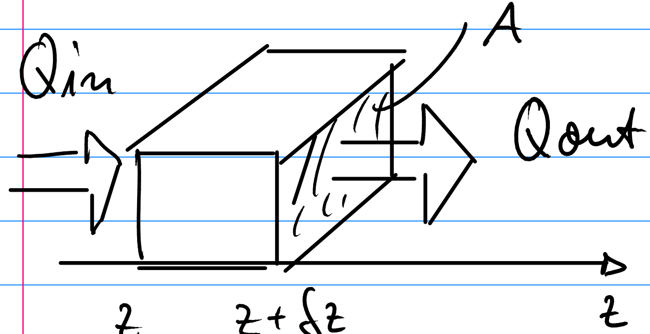
$$\underline{q} = -k \, \underline{\nabla} T$$ Fick's law

$[q] = \frac{W}{m^3}$ $[k] = \frac{W}{mK}$

$= \frac{J}{s m^3}$

k = conductivity $\sim 4 \, \frac{W}{mK}$ for rocks

Consider total heat energy change in volume



$Q = q \cdot A$ $[Q] = W$

$= \frac{\text{change of heat}}{\text{time}}$

$$Q_{out} = Q_{in} + \frac{\partial Q}{\partial z} \delta z$$

$$\Delta Q_g = Q_{in} - Q_{out} = -\delta z \frac{\partial Q}{\partial z}$$

change due to gradient in heat flow

Consider internal sources

$$\Delta Q_i = H \cdot A \cdot \delta z = H \cdot V \quad [H] = \frac{W}{m^3}$$

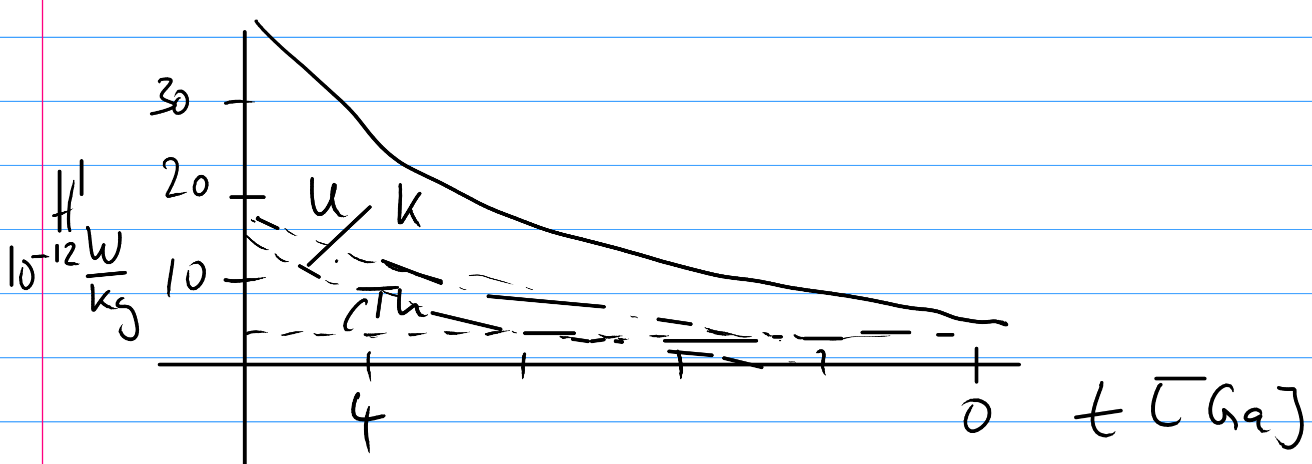
$$H = \frac{\text{internal heat production}}{\text{unit time} \times \text{unit volume}}$$

In the Earth's mantle due to U, K (Ga half life) and Th (10 Ga half-life).

H in continents is $\sim 2.6\ \mu W/m^3$. Often, H is referred to in units of

$$H' = H/\rho \; ; \; [H'] = \frac{W}{kg}$$

i.e. heat rate per unit mass. Then, $H' \sim nW/kg$, for continents. The upper mantle has $H \sim 3\ nW/m^3$ (less than 1/1000 of continental crust), and chondritic meteorites $\sim 16\ nW/m^3$. (Note that internal heat was $\sim$ 4-5 times higher 4.6 Ga ago.



Total energy change due to heat can be written

$$\Delta E = m \cdot c_p \cdot \Delta T \qquad [c_p] = \frac{J}{kg \cdot K}$$

$$c_p = 1000 \ \frac{J}{kg\,K} \text{ for rocks}$$

per time
$$\frac{\Delta E}{\Delta t} = m \cdot c_p \frac{\Delta T}{\Delta t} = Q = Q_i + Q_g$$

$$\rho A \delta z \, c_p \frac{\partial T}{\partial t} = -\delta z \frac{\partial Q}{\partial z} + H \cdot A \cdot \delta z$$

$$Q = Aq = -kA \frac{\partial T}{\partial z}$$

$$\leadsto \quad \rho c_p \frac{\partial T}{\partial t} = kA \frac{\partial^2 T}{\partial z^2} + H$$

or
$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial z^2} + \frac{H}{\rho c_p}$$

$$\varkappa = \frac{k}{\rho c_p} \quad \text{thermal diffusivity} \quad [\varkappa] = \frac{m^2}{s}$$

$$\varkappa \sim 10^{-6} \ \frac{m^2}{s} \text{ for rocks}$$

3D conduction eq.

$$\frac{\partial T}{\partial t} = \varkappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{H}{\rho c_p} = \varkappa \nabla^2 T + \frac{H}{\rho c_p}$$

For transport problems, consider Eulerian frame where

$$\frac{\partial}{\partial t} \longrightarrow \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}$$

i.e. $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z} = \kappa \nabla^2 T + \frac{H}{\rho c_p}$

(simplified "energy equation"). For now consider $\underline{u} = \underline{0}$, pure conduction.

<u>Steady state solutions</u> (geotherms)

$$\nabla^2 T = -\frac{H}{k}$$

$$\frac{\partial^2 T}{\partial z^2} = -\frac{H}{k} \rightsquigarrow \frac{\partial T}{\partial z} = -\frac{H}{k} z + c_1 \rightsquigarrow T(z) = -\frac{H}{2k} z^2 + c_1 z + c_2$$

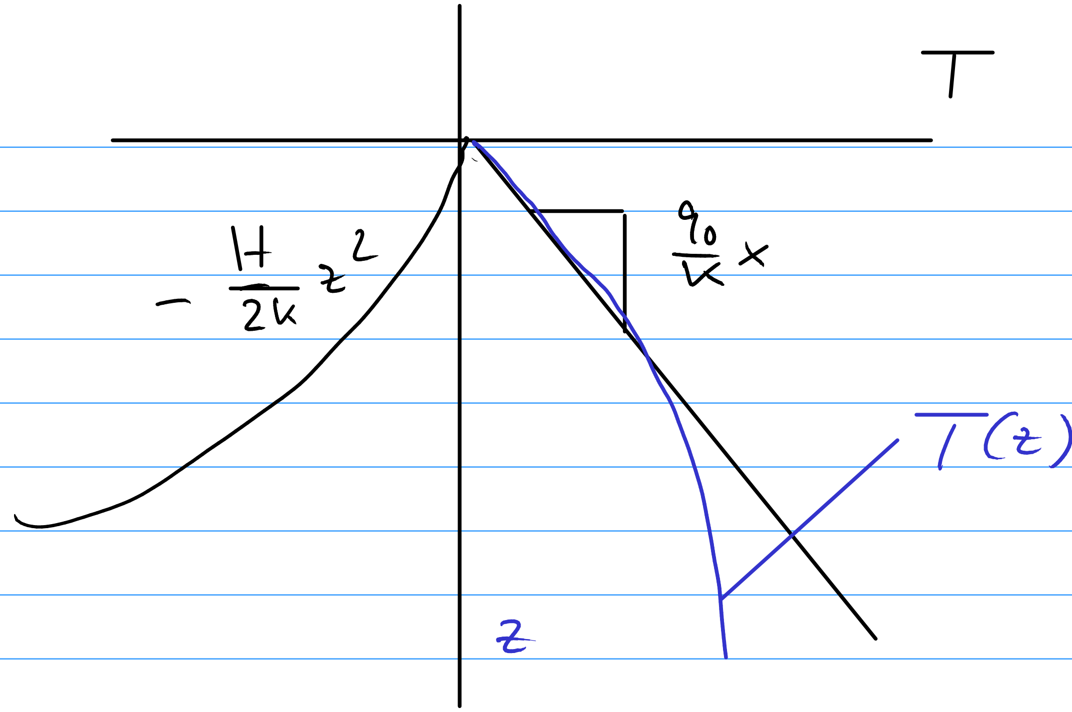
2^{nd} order PDE $\rightsquigarrow$ need two BCs

<u>Example 1</u> (1) $T(z=0) = 0$ (problem linear, can always add T_0)

(2) $q(z=0) = -q_0$ (positive out of Earth)

(1) $\rightsquigarrow$ $c_2 = 0$ (2) $q = -k \frac{\partial T}{\partial z} \rightsquigarrow c_1 = \frac{q_0}{k}$

$$\rightsquigarrow T(z) = -\frac{H}{2k} z^2 + \frac{q_0}{k} z$$



<u>Example 2</u>

(1) $T(z=0)=0$ (2) $q(z=d) = -q_d$

$$\rightsquigarrow \quad T(z) = -\frac{H}{2k} z^2 + \frac{q_d \cdot Hd}{k} z$$

Compare with example (1) $\rightsquigarrow$ $q_0 = q_d + H \cdot d$

surface heat flux = bottom heat flux plus radioactive contrib.

<u>Example 3</u>

(1) $T(z=0)=0$ (2) $q(z=\infty) = -q_m$

and $H(z) = H_0 \exp(-z/d)$

$$\frac{\partial^2 T}{\partial z^2} = -\frac{H_0 \exp(-z/d)}{k}$$

$\times (-k)$ integrate

$$-k \frac{\partial T}{\partial z} = -d H_0 e^{-\frac{z}{d}} + c_1$$

(2) $c_1 = -q_m \rightsquigarrow q(z) = -k \frac{\partial T}{\partial z} = -q_m - d H_0 e^{-\frac{z}{d}}$

$-q(z=0) = q_m + d H_0 \rightsquigarrow$ observed

Example 4

Steady state solution of an internally heated sphere (Turcotte & Schubert, 4-9)

$$T = -\frac{H}{6k} r^2 + \frac{c_1}{r} + c_2$$

for $T(0)=0$ and finite T at $r=0$ so

$$T(r) = \frac{H}{6k}(a^2 - r^2)$$

$$q_0 = \frac{Ha}{3} \quad \text{with } a \text{ radius of sphere}$$

Time-dependent heat conduction

1D
$$\frac{\partial T}{\partial t} = \varkappa \frac{\partial^2 T}{\partial z^2} \qquad (1)$$

Example 1

Periodic heating of a semi-infinite half space

(2) $T_s = T_0 + \Delta T \cos \omega t \qquad \omega = 2\pi\nu = \frac{2\pi}{\tau}$

Use separation of variables to solve (1)

$$T(z,t) = T_0 + A(z)\,B(t)$$

assume harmonic solution

$$= T_0 + Z_1(z) \cos \omega t + Z_2 \sin \omega t$$

Plug into (1) $\quad \partial_t T = -Z_1 \omega \sin \omega t + Z_2 \omega \cos \omega t$

$$\partial_{zz} T = \partial_{zz} Z_1 \cos \omega t + \partial_{zz} Z_2 \sin \omega t$$

$$\rightsquigarrow \quad -Z_1 \omega = \varkappa \frac{d^2 Z_2}{dz^2} \quad ; \quad Z_2 \omega = \varkappa \frac{d^2 Z_1}{dz^2} \qquad (3)$$

$$\rightsquigarrow \quad \frac{d^4 Z_2}{dy^4} + \frac{\omega^2}{\varkappa^2} Z_2 = 0 \qquad (\text{same for } Z_1)$$

Assume Z_2 is of the form $Z_2 = c\, e^{\alpha z}$

$$\leadsto \quad \alpha^4 + \frac{\omega^2}{\varkappa^2} = 0$$

$$ax^2 + bx + c = 0$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$i = \sqrt{-1}$$

$$\beta = \alpha^2 \leadsto \beta = \pm \frac{\sqrt{-\frac{4\omega^2}{\varkappa^2}}}{2} = \pm i \frac{\omega}{\varkappa}$$

$$\alpha^2 = \pm i \frac{\omega}{\varkappa}$$

$$x^2 = i \leadsto x = \frac{1 \pm i}{\sqrt{2}}$$

$$\leadsto \quad \alpha = \pm \left(\frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{\varkappa}}$$

$$t_2 = c_1 \exp\left(\frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega}{\varkappa}}\, z \right) + c_2 \exp\left(\frac{1-i}{\sqrt{2}} \sqrt{\frac{\omega}{\varkappa}}\, z \right) +$$
$$c_3 \exp\left(-\frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega}{\varkappa}}\, z \right) + c_4 \exp\left(-\frac{1-i}{\sqrt{2}} \sqrt{\frac{\omega}{\varkappa}}\, z \right)$$

Since temperature fluctuations have to decay with depth $c_1 = c_2 = 0$, can rewrite as

$$Z_2 = \exp(-kz)\left(b_1 \cos kz + b_2 \sin kz\right)$$

Because $e^{i\theta} = \cos\theta + i\sin\theta$ as $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$; $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

with $k = \sqrt{\frac{\omega}{2\varkappa}}$. Likewise for

$$Z_1 = \exp(-kz)\left(b_3 \cos kz + b_4 \sin kz\right)$$

For (3) to hold, $b_2 = b_3$ and $b_1 = -b_4$.

For (2) to apply at the surface, $b_1 = 0$ and $b_3 = \Delta T$

$\rightsquigarrow T(z,t) = T_0 + \Delta T \exp\left(-kz\right) \cos\left(\omega t - zk\right)$

Amplitude decays with $1/k$, the skin depth

$$d_S = \sqrt{\frac{2\varkappa}{\omega}} \quad \text{(note dimensional argument)}$$

With diurnal variations, $\omega = 7.27 \cdot 10^{-5} \frac{rad}{s}$, and with $\varkappa = 10^{-6} \frac{m^2}{s}$, $d_S \sim 0.17m$.

Note shift in phase with depth

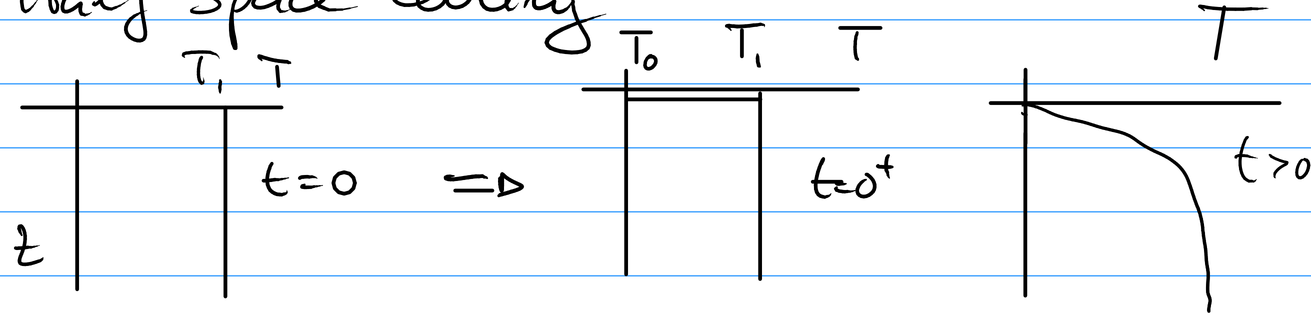
$$\phi = \frac{z}{d_S}$$

i.e. $T(z)$ will be out of phase with the surface at $\phi = \pi \rightsquigarrow z_\pi = \pi d_S$, i.e. $\sim 0.5m$ for a diurnal signal.

Example 2

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

Half-space cooling



BC and ICs

$$T = T_1 \quad \text{at } t=0$$
$$T = T_0 \quad \text{at } z=0 \text{ for } t>0$$
$$T = T_1 \quad \text{at } z=\infty$$

Introduce dimensionless temperature

$$\theta = \frac{T - T_1}{T_0 - T_1} \quad \text{for which} \quad \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2} \qquad (1)$$

BCs and ICs

$$\theta(z,t) = 0$$

$\theta(z,t)$

$$\theta(0,t) = 1 \qquad (2)$$
$$\theta(\infty,t) = 0$$

Solve (1) by introduction of a similarity variable

$$\eta = \frac{z}{2\sqrt{\kappa t}}$$

which combines the effect of space and time.

Then

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial t} = \frac{d\theta}{d\eta}\left(-\frac{z}{4\sqrt{\kappa}} t^{-\frac{3}{2}}\right) = \frac{d\theta}{d\eta}\left(-\frac{1}{2}\frac{\eta}{t}\right)$$

$$\frac{\partial \theta}{\partial z} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial z} = \frac{d\theta}{d\eta} \quad \frac{1}{2\sqrt{\kappa t}}$$

$$\frac{\partial^2 \theta}{\partial z^2} = \frac{1}{2\sqrt{\kappa t}} \quad \frac{d^2\theta}{d\eta^2}\frac{\partial \eta}{\partial z} = \frac{1}{4\kappa t}\frac{d^2\theta}{d\eta^2}$$

(1) then becomes

$$-\eta \frac{d\theta}{d\eta} = \frac{1}{2}\frac{d^2\theta}{d\eta^2} \qquad (3)$$

Note that both $z=\infty$ and $t=0$ map to $\eta=\infty$
i.e. BCs (2) map to

$$\theta(\infty) = 0$$
$$\theta(0) = 1$$

and the PDE (1) has been reduced to an ODE (3),
which can be solved by

$$\phi = \frac{d\theta}{d\eta}$$

$$-\eta\phi = \frac{1}{2}\frac{d\phi}{d\eta} \rightsquigarrow -\eta \, d\eta = \frac{1}{2}\frac{d\phi}{\phi}$$ $\swarrow$ int.

$$-\eta^2 = \ln\phi - \ln c_1$$ $\swarrow$

$$\phi = c_1 e^{-\eta^2} = \frac{d\theta}{d\eta}$$ $\swarrow$ integrate using $\theta(0)=1$

$$\theta = c_1 \int_0^{\eta} e^{-\eta'^2} d\eta' + 1$$

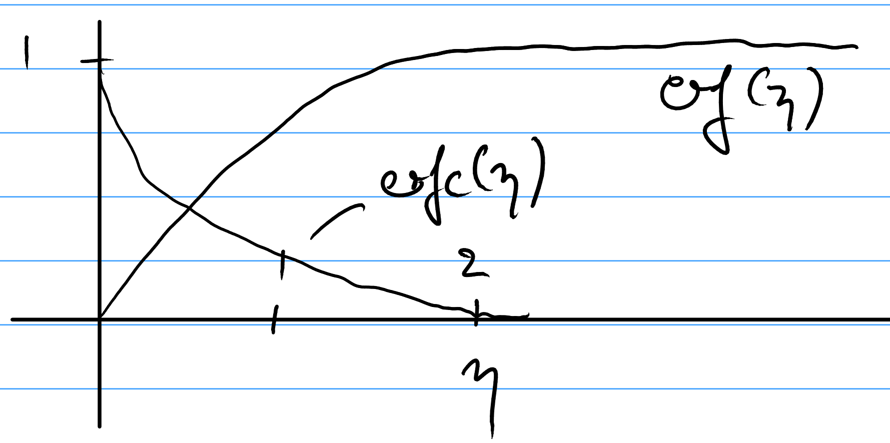
For $\theta(\infty) = 0$ $\quad c_1 \underbrace{\int_0^\infty e^{-\eta'^2} d\eta'}_{= \sqrt{\pi}/2} + 1 = 0$

$$\leadsto \theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta'$$

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta' \qquad \text{error function}$$

$$\leadsto \theta = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}\,\eta \qquad \text{complement. error function}$$

$$\leadsto \boxed{\frac{T - T_1}{T_0 - T_1} = \mathrm{erfc}\left(\frac{z}{2\sqrt{\varkappa t}}\right)} \quad (4)$$



$\leadsto$ Thermal boundary layer, define thickness as $z\,(\theta = 0.1)$ $\leadsto$

$$\eta_T = \mathrm{erfc}^{-1}\, 0.1 = 1.16$$

$$\leadsto z_T = 2\eta_T \sqrt{\varkappa t} = 2.32\sqrt{\varkappa t}$$

Heat flux

$$q = -k \frac{\partial T}{\partial z} \quad (4)$$

$$q = -k\,(T_0 - T_1)\,\partial_z \operatorname{erfc}\left(\frac{z}{2\sqrt{\varkappa t}}\right)$$

$$= k\,(T_0 - T_1)\,\partial_z \operatorname{erf}\left(\frac{z}{2\sqrt{\varkappa t}}\right)$$

$$\eta = \frac{z}{2\sqrt{\varkappa t}} \quad ; \quad \frac{d\eta}{dz} = \frac{1}{2\sqrt{\varkappa t}}$$

$$\frac{df}{dz} = \frac{df}{d\eta}\frac{d\eta}{dz}$$

$$\rightsquigarrow \quad q = \frac{k\,(T_0 - T_1)}{2\sqrt{\varkappa t}}\,\partial_\eta \operatorname{erf}(\eta)$$

$$= \frac{k\,(T_0 - T_1)}{\sqrt{\pi \varkappa t}}\, e^{-\eta^2}$$

Heat flux at the surface $\quad \eta = 0$

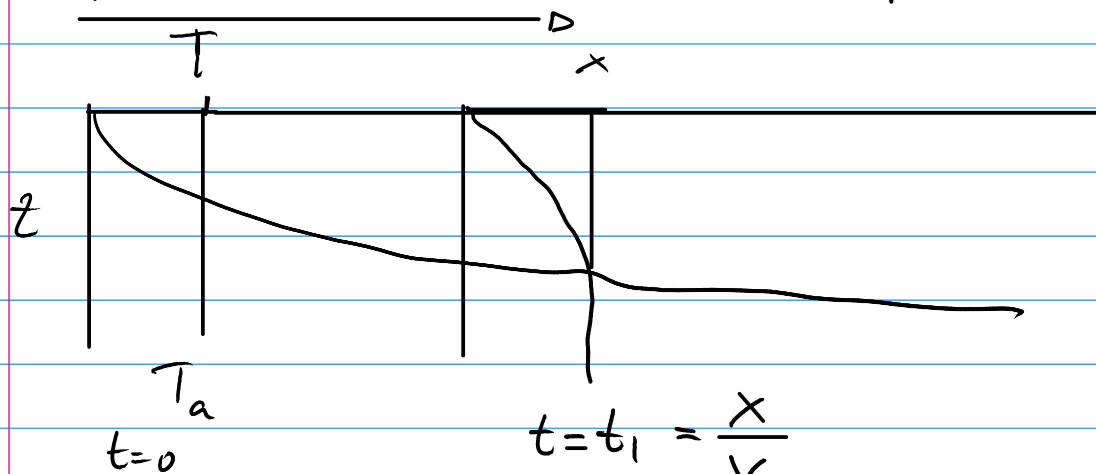
$$q = \frac{k\,(T_0 - T_1)}{\sqrt{\pi \varkappa}} \frac{1}{\sqrt{t}}$$

Kelvin's estimate of the age of the Earth

$$\sqrt{t} \simeq \frac{k\,(T_0 - T_1)}{\sqrt{\pi\varkappa}\, k \frac{\partial T}{\partial z}} \rightsquigarrow t = \frac{(T_0 - T_1)^2}{\pi \varkappa\, (\partial_z T)^2}$$

With $\varkappa = 10^{-6}\,\frac{m^2}{s}$, $\Delta T = 2000\,°K$, $\partial_z T = 25\,\frac{K}{km}$ $\rightsquigarrow t = 65\,Myr.$

Application to oceanic lithosphere



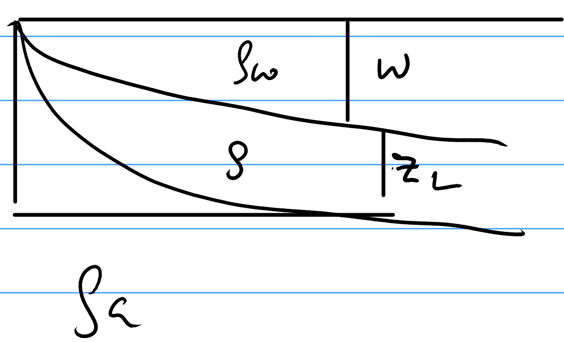
$$t = t_1 = \frac{x}{v}$$

$\rightsquigarrow T_0 = 0 \quad T_1 = T_a$

$$T = T_a \operatorname{erf}\left(\frac{z}{2\sqrt{\kappa\, age}} \right) = T_a \operatorname{erf}\left(\frac{z}{2\sqrt{\kappa x / v}} \right)$$

$$q = \frac{k T_a}{\sqrt{\pi \kappa}} \frac{1}{\sqrt{age}} = \frac{k T_a}{\sqrt{\pi \kappa}} \sqrt{\frac{v}{\pi \kappa x}}$$

Ocean floor topography



isostasy

$$\rho_a (w + z_L) = \rho_w w + \int_0^{z_L} \rho \, dz$$

$$(\rho_a - \rho_w) w = \int_0^{z_L} (\rho - \rho_a) \, dz$$

$$\rho - \rho_a = \rho_a \alpha (T_a - T) \quad \text{because } \Delta\rho = \rho_0 \alpha \Delta T$$

$$w(\rho_a - \rho_w) = \rho_a \alpha (T_a - T_S) \int_0^{\infty} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}} \right) dz$$

where z_L has been replaced with ∞ as $T \to T_a$ at the base of the lithosphere.

With

$$\eta = \frac{z}{2\sqrt{\kappa t}} \qquad \frac{1}{\sqrt{\pi}}$$

$$w = \frac{2\rho_a \alpha (T_a - T_0)}{\rho_a - \rho_w} \sqrt{\kappa t} \int_0^{\infty} \operatorname{erfc}(\eta)\, d\eta$$

$$\boxed{w = \frac{2\rho_a \alpha (T_a - T_0)}{\rho_a - \rho_w} \sqrt{\frac{\kappa t}{\pi}}} \; ; \quad t = \frac{x}{v}$$