Heat Transport

- radiation
- conduction
- convection

[combine to get convection]

Conduction

Heat is transported along gradients of temperature.

\[ q \propto \frac{T_2 - T_1}{z_2 - z_1} = \frac{\Delta T}{\Delta z} \]

\[ q = \frac{W}{m^3} \quad [k] = \frac{W}{m \cdot K} \quad q = -k \nabla T \quad \text{Fick's law} \]

\[ k = \text{conductivity} = 4 \times 10^6 \text{ J/m \cdot K} \] for rocks

Consider total heat energy change in volume

\[ Q_{\text{in}} \]

\[ Q_{\text{out}} \]

\[ Q_{\text{in}} - Q_{\text{out}} = -\int_{z_1}^{z_2} \frac{\partial Q}{\partial t} \, dz \]

\[ \Delta Q_g = Q_{\text{in}} - Q_{\text{out}} = -\int_{z_1}^{z_2} \frac{\partial Q}{\partial t} \, dz \]

change due to gradient in heat flow
Consider internal sources

\[ \Delta Q_i = H \cdot A \cdot \Delta t = H \cdot V \left[ \frac{W}{m^3} \right] \]

\[ H = \text{internal heat production} \]

unit time \times \text{unit volume}

In the Earth's mantle due to U, k (6 Ga half-life) and Th (10 Ga half-life).

\( H \) in continents is \( \approx 2.6 \mu W/m^3 \). Often, \( H \) is referred to in units of

\[ H' = \frac{H}{S} ; \left[ H' \right] = \frac{W}{kg} \]

i.e. heat rate per unit mass. Then, \( H' \approx h W/kg \) for continents. The upper mantle has \( H \approx 3 W/m^3 \) (less than \( 1/1000 \) of continental crust), and chondritic meteorites \( \approx 16 W/m^3 \). (Note that internal heat was \( \approx 4-5 \) times higher \( 4.6 \) Ga ago.

![Graph showing the decay of heat production over time.](image-url)
Total energy change due to heat can be with

\[ \Delta E = m \cdot c_p \cdot \Delta T \]

\[ c_p = \frac{\text{J}}{\text{kg} \cdot \text{K}} \]

\[ c_p = 1000 \frac{\text{J}}{\text{kg} \cdot \text{K}} \text{ rocks} \]

per time

\[ \frac{\Delta E}{\Delta t} = m \cdot c_p \frac{\Delta T}{\Delta t} = Q = Q_i + Q_g \]

\[ 8 ASD \cdot c_p \frac{\partial T}{\partial t} = - \delta z \frac{\partial \phi}{\partial t} + \mathbf{H} \cdot \mathbf{A} \cdot \delta z \]

\[ Q = Aq = -kA \frac{\partial T}{\partial t} \]

\[ 8A \cdot c_p \frac{\partial T}{\partial t} = kA \frac{\partial^2 T}{\partial z^2} + H \]

\[ \sigma \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} + \frac{H}{Sc_p} \]

\[ \alpha = \frac{k}{Sc_p} \text{ thermal diffusivity} \]

\[ \alpha = \frac{u^2}{s} \approx 10^{-6} \frac{u^2}{s} \text{ for rocks} \]

3D conduction eq.

\[ \frac{\partial^2 T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{H}{Sc_p} = \alpha \nabla^2 T + \frac{H}{Sc_p} \]
For transport problems, consider Eulerian frame where
\[
\frac{\partial}{\partial t} \rightarrow \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla
\]
i.e.
\[
\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z} = u \cdot \nabla T + \frac{H}{\rho c_p}
\]
(simplified “energy equation”). For now, consider \( u = 0 \) (pure conduction).

Steady state solutions (geotherms)

\[
\nabla^2 T = \frac{-H}{K}
\]

\[
\frac{\partial^2 T}{\partial z^2} = \frac{-H}{K} \quad \Rightarrow \quad \frac{\partial T}{\partial z} = \frac{-H}{K} z + c_1 \quad \Rightarrow \quad T(z) = \frac{-H}{2K} z^2 + c_1 z + c_2
\]

2nd order PDE needs two BCs

Example (1) \( T(z=0) = 0 \) (problem linear, can always add \( T_0 \))

(2) \( q(z=0) = -q_0 \) (positive out of Earth)

\( \Rightarrow c_2 = 0 \)

(2) \( q = -K \frac{\partial T}{\partial z} \Rightarrow c_1 = \frac{q_0}{K} \)

\( \Rightarrow T(z) = -\frac{H}{2K} z^2 + \frac{q_0}{K} z \)
Example 2

(1) \( T(z=0) = 0 \)
(2) \( q(z=d) = -q_0 \)

\[ T(z) = -\frac{H}{2k} z^2 + \frac{q_0 \cdot H \cdot d}{k} \]

Compare with example (1) \( \Rightarrow q_0 = q_0 \cdot d + H \cdot d \)

Surface heat flux = bottom heat flux plus radioactive contrib.

Example 3

(1) \( T(z=0) = 0 \)
(2) \( q(z=\infty) = -q_w \)

and \( H(z) = H_0 \exp(-z/d) \)

\[ \frac{\partial^2 T}{\partial z^2} = -\frac{H_0 \exp(-z/d)}{k} \]

integrate \(-k \frac{\partial T}{\partial z} = -dH_0 e^{-z/d} + c_1 \)

(2) \( c_1 = -q_w \Rightarrow q(z) = -k \frac{\partial T}{\partial z} = -q_w - dH_0 e^{-\frac{z}{d}} \)

\( -q(z=0) = q_w + dH_0 \) not observed
Example 4

Steady state solution of an internally heated sphere (Twomey & Schmidt, 4-9)

\[ T = -\frac{H}{6k} r^2 + \frac{c_1}{r} + c_2 \]

for \( T(0) = 0 \) and finite \( T \) at \( r = 0 \) \( \rightarrow \)

\[ T(r) = \frac{H}{6k} (a^2 - r^2) \]

\[ q_0 = \frac{Ha^2}{3} \quad \text{with a radius of sphere} \]
Time-dependent heat conduction

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \]  \hspace{1cm} (1)

Example 1

Periodic heating of a semi-infinite half space

\[ T_s = T_0 + \Delta T \cos \omega t \quad \omega = \frac{2\pi}{t} \]

Use separation of variables to solve (1)

\[ T(z,t) = T_0 + A(z) B(t) \]

solutions

\[ T(z,t) = T_0 + z_1(z) \cos \omega t + z_2(z) \sin \omega t \]

Plug into (1)

\[ \frac{\partial T}{\partial t} = -z_1 \omega \sin \omega t + z_2 \omega \cos \omega t \]

\[ \frac{\partial^2 T}{\partial z^2} = \frac{\partial^2 z_1}{\partial z^2} \cos \omega t + \frac{\partial^2 z_2}{\partial z^2} \sin \omega t \]

\[ -\omega^2 z_1 = \alpha \frac{\partial^4 z_1}{\partial z^4} \quad ; \quad -\omega^2 z_2 = \alpha \frac{\partial^4 z_2}{\partial z^4} \]  \hspace{1cm} (3)

Assume \( z_2 \) is of the form \( z_2 = ce^{zt} \)
\[ \alpha^4 + \frac{\omega^2}{\alpha^2} = 0 \]
\[ \alpha = \pm \frac{\sqrt{4\omega^2 - 4\omega^2}}{2} = \pm i \frac{\omega}{\omega} \]
\[ \beta = \alpha^2 = \pm i \frac{\omega}{\omega} \]
\[ x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
\[ \omega = i \]
\[ x = \frac{1 \pm i}{\sqrt{2}} \]
\[ \alpha = \pm \left( \frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{\omega^2}} \]
\[ z_2 = c_1 \exp \left( \frac{1 + i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha^2}} \right) + c_2 \exp \left( \frac{1 - i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha^2}} \right) + c_3 \exp \left( - \frac{1 + i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha^2}} \right) + c_4 \exp \left( - \frac{1 - i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha^2}} \right) \]

Since temperature fluctuations have to decay with depth, \( c_1 = c_2 = 0 \), can rewrite as
\[ z_2 = \exp(-kz) \left( b_1 \cos kz + b_2 \sin kz \right) \]

Because \( e^{i\theta} = \cos \theta + i \sin \theta \) as \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \) and \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \)

with \( k = \sqrt{\frac{\omega}{2\alpha}} \). Likewise for
\[ z_1 = \exp(-kz) \left( b_3 \cos kz + b_4 \sin kz \right) \]

For (3) to hold, \( b_2 = b_3 \) and \( b_1 = -b_4 \).

For (2) to apply at the surface, \( b_1 = 0 \) and \( b_3 = \Delta T \).
\[ T(z,t) = T_0 + \Delta T \exp(-kz) \cos(\omega t - zk) \]

Amplitude decays with \( 1/k \), the skin depth

\[ d_s = \sqrt{\frac{2 \nu}{\omega}} \]  

(note dimensional argument)

With diagonal variations, \( \omega = 7.27 \cdot 10^{-5} \ \text{rad} \ \text{s}^{-1} \), and

with \( \nu = 10^{-6} \ \text{m}^2 \ \text{s}^{-1} \), \( d_s \approx 0.17 \ \text{m} \).

Note shift in phase with depth

\[ \phi = \frac{\pi}{d_s} \]

i.e. \( T(z) \) will be out of phase with the surface

at \( \phi = \pi \) \( \Rightarrow \) \( z = \frac{\pi}{\omega} d_s \), i.e. \( \approx 0.5 \ \text{m} \) for a diagonal signal.
Example 2

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

Half-space cooling

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

BC and ICs

\[ T=T_1 \quad \text{at} \quad t=0 \]
\[ T=T_0 \quad \text{at} \quad x=0 \quad \text{for} \quad t > 0 \]
\[ T=T_1 \quad \text{at} \quad x=\infty \]

Introduce dimensionless temperature

\[ \Theta = \frac{T-T_1}{T_0-T_1}, \quad \text{for which} \quad \frac{\partial \Theta}{\partial t} = \alpha \frac{\partial^2 \Theta}{\partial x^2} \]

BCs and ICs

\[ \Theta (x, t)=0 \quad \text{(1)} \]
\[ \Theta (0, t)=1 \quad \text{(2)} \]
\[ \Theta (\infty, t)=0 \]

Solve (1) by introduction of a similarity variable

\[ \eta = \frac{x}{2 \sqrt{\alpha t}} \]

which combines the effect of space and time.

Then

\[ \frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial \Theta}{\partial \eta} \left( \frac{x}{4 \alpha t} t^{-2} \right) = \frac{\partial^2 \Theta}{\partial \eta^2} \left( \frac{1}{2} \frac{1}{t} \right) \]
\[ \frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial z} = \frac{d\theta}{d\eta} \frac{1}{2\alpha e^t} \]

\[ \frac{d^2 \theta}{dz^2} = \frac{1}{2\alpha e^t} \frac{d^2 \theta}{d\eta^2} \frac{\partial \eta}{\partial z} = \frac{1}{4\alpha t} \frac{d^2 \theta}{d\eta^2} \]

(1) then becomes

\[ -\eta \frac{d\theta}{d\eta} = \frac{1}{2} \frac{d^2 \theta}{d\eta^2} \quad (3) \]

Note that both \( z = \infty \) and \( t = 0 \) map to \( \eta = \infty \)

i.e. BCS (2) map to

\[ \theta(\infty) = 0 \]

\[ \theta(0) = 1 \]

and the PDE (1) has been reduced to an ODE (3), which can be solved by

\[ \phi = \frac{d\theta}{d\eta} \]

\[ -\eta \phi = \frac{1}{2} \frac{d\phi}{d\eta} \quad \Rightarrow \quad -\eta d\eta = \frac{1}{2} \frac{d\phi}{\phi} \quad \text{integrate} \]

\[ -\eta^2 = \ln \phi - \ln c_1, \quad 2 \]

\[ \phi = c_1 e^{-\eta^2} = \frac{d\theta}{d\eta} \quad \text{integrate using } \theta(0) = 1 \]

\[ \theta = c_1 \int_{0}^{\eta} e^{-\eta_1^2} \, d\eta_1 + 1 \]
For \( \theta(\infty) = 0 \)
\[
\int_0^\infty e^{-\eta^2} d\eta / \sqrt{\pi} + 1 = 0
\]
\[
\Rightarrow \theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta'
\]

\( \text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta'^2} d\eta' \) function

\( \Rightarrow \theta = 1 - \text{erf}(\eta) = \text{erfc}(\eta) \) complement,

\( \text{erf}(\eta) \) function

\[
\frac{T - T_1}{T_0 - T_1} = \text{erfc} \left( \frac{z}{2 \sqrt{\alpha t}} \right)
\]

\[
\text{erf}(\eta)
\]

Thermal boundary layer, define thickness as \( \eta_T \) (\( \theta = 0.1 \))

\[
\eta_T = \text{erfc}^{-1}(0.1) = 1.16
\]

\[
\Rightarrow \xi_T = 2\eta_T \sqrt{\alpha t} = 2.32 \sqrt{\alpha t}
\]
Heat flux

\[ q = -k \frac{\partial T}{\partial z} \]  \( \beta \)  (4)

\[ q = -k (T_0 - T_1) \frac{\partial T}{\partial z} \text{ erf} \left( \frac{z}{2 \sqrt{\pi \alpha t}} \right) \]

\[ = k (T_0 - T_1) \frac{\partial T}{\partial z} \text{ erf} \left( \frac{z}{2 \sqrt{\pi \alpha t}} \right) \]

\[ \eta = \frac{z}{2 \sqrt{\pi \alpha t}} \quad ; \quad \frac{d\eta}{dz} = \frac{1}{2 \sqrt{\pi \alpha t}} \]

\[ \frac{d}{dt} = \frac{d}{d\eta} \frac{d\eta}{dz} \]

\[ \sim \quad q = \frac{k (T_0 - T_1)}{2 \sqrt{\pi \alpha t}} \eta \text{ erf} (\eta) \]

\[ = \frac{k (T_0 - T_1)}{\sqrt{\pi \alpha t}} e^{-\eta^2} \]

Heat flux at the surface \( \eta = 0 \)

\[ q = \frac{k (T_0 - T_1)}{\sqrt{\pi \alpha t}} \frac{1}{\sqrt{t}} \]

Kelvin's estimate of the age of the Earth

\[ \sqrt{t} = \frac{k (T_0 - T_1)}{\sqrt{\pi \alpha k} \frac{\partial T}{\partial z}} \sim t = \frac{(T_0 - T_1)^2}{\pi \alpha C_0^2 T^2} \]

With \( \alpha = 10^{-6} \text{ m}^2 / \text{s} \), \( \Delta T = 2000^\circ \text{K} \), \( \partial z = 25 \frac{\text{K}}{\text{m}} \) \( \Rightarrow t = 65 \text{Myr} \).
**Application to oceanic lithosphere**

\[
T = T_a \left(1 - \frac{2}{\sqrt{\pi} \sigma} \right) = \frac{T_a}{\sqrt{\pi \sigma}} \sqrt{\frac{v}{\pi \sigma}}
\]

\[
q = \frac{kT_a}{\sqrt{\pi \sigma C_T}} = \frac{kT_a}{\sqrt{\pi \sigma C_T}} \sqrt{\frac{v}{\pi \sigma}}
\]

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**Ocean floor topography**

Isostasy

\[
S_a (w + z_L) = S_w + \int_0^{z_L} S \, dz
\]

\[
(S_a - S_w) w = \int_0^{z_L} (S - S_a) \, dz
\]

\[
S - S_a = S_a \alpha (T_a - T)
\]

\[
w (S_a - S_w) = S_a \alpha (T_a - T_s) \int_0^{z_L} e^{-\frac{t^2}{2\pi \sigma}} \, dt
\]
where \( z_1 \) has been replaced with \( \infty \) as \( T \to T_0 \) at the base of the lithosphere.

With

\[
\eta = \frac{z}{2\sqrt{\pi t}}
\]

\[
\omega = \frac{2S_0 \chi (T_a - T_0)}{S_a - S_0} \int_0^\infty \text{erfc} (\eta) d\eta
\]

\[
\omega = \frac{2S_0 \chi (T_a - T_0)}{S_a - S_0} \sqrt{\frac{\pi t}{2}}
\]

; \quad t = \frac{x}{v}