534 Fluid dynamics and thermal convection

Consider fluid flow

$$\tau = 2\eta\dot{\varepsilon}$$

with η the dynamic viscosity with units force/area/second. Often, useful to consider

$$\nu = \frac{\eta}{\rho}$$

the kinematic viscosity with units length^2/time, i.e. a diffusivity. Can define the Prandtl number as

$$Pr = \frac{\nu}{\kappa}$$

to measure the strength of momentum diffusion to thermal diffusion. For the mantle,

$$Pr \sim \frac{10^{21}}{3 \cdot 10^{3}\ 10^{-6}} = \frac{1}{3}\ 10^{24}$$

i.e. momentum diffuses much more rapidly than heat, which allows us to neglect the role of inertia in thermally driven flow (the infinite Prandtl ~~#~~ assumption).

Can also define the Reynolds number,

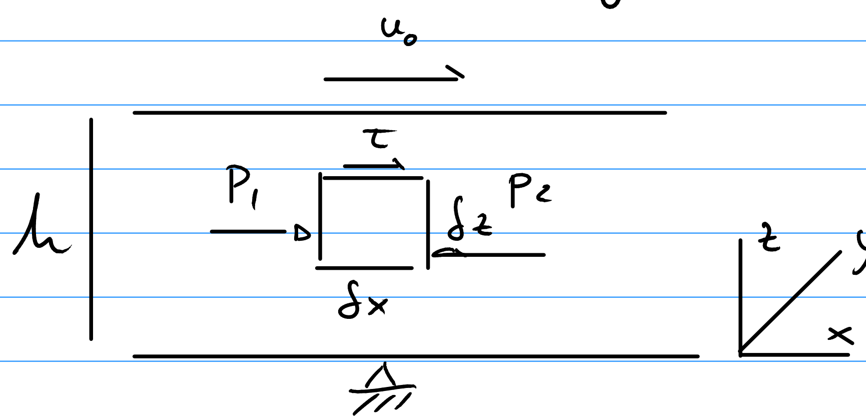
$$Re = \frac{\rho v L}{\eta}$$

where $Re \simeq 0$ denotes laminar and $Re \gg 1$ turbulent flow. For the mantle — large L!

$$Re \simeq \frac{3\cdot 10^3 \; 10^{-1} \; 3\cdot 10^6}{\pi \cdot 10^7 \; 10^{21}} \simeq 3\cdot 10^{-21}$$

$\rightsquigarrow$ mantle flow is clearly in the laminar regime.

Consider channel flow



channel is infinite in y direction

net pressure force on fluid element in x direction

$$F_p = (p_2 - p_1)\,\delta z$$

shear force on top minus shear force in bottom

$$F_s = \left(\tau + \frac{d\tau}{dz}\delta z\right)\delta x - \tau\,\delta x = \frac{d\tau}{dz}\delta z\,\delta x$$

forces have to balance

$$(p_2 - p_1)\,\delta z = \frac{d\tau}{dz}\delta z\,\delta x$$

$$\frac{p_2 - p_1}{\delta z} = \frac{d\tau}{dz} \rightsquigarrow \frac{d\tau}{dz} = \frac{dp}{dx}$$

$$\tau = 2\eta \dot{\varepsilon}_{xz} = \eta \frac{du}{dz}$$

$$\rightarrow \quad \eta \frac{d^2u}{dz^2} = \frac{dp}{dx} \qquad (1)$$

In general, for 2D and Newtonian viscosity

$$-\underline{\nabla} p + \eta \nabla^2 \underline{v} = \underline{F}_b$$

If $\underline{F}_b = -\rho g \underline{e}_z$ $\rightarrow$ $\quad \underline{v} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

$$-\frac{\partial p}{\partial x} + \eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

$$-\frac{\partial p}{\partial z} + \eta \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) = -\rho g$$

"Stokes equation" for Newtonian viscosity, force balance equation, or conservation of momentum.

<u>Continuity equation</u> for incompressible flow

$$\underline{\nabla} \cdot \underline{v} = 0$$

or
$$\frac{\partial v}{\partial z} + \frac{\partial u}{\partial x} = 0 \qquad (2)$$

Useful to define a streamfunction Ψ such that

$$u = -\partial_z \Psi \quad \text{and} \quad v = \partial_x \Psi$$

which naturally satisfies incompressibility.

Solutions to channel flow

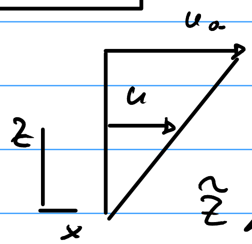
$$\eta \frac{\partial^2 u}{\partial z^2} = \frac{dp}{dx}$$

$$u(z) = \frac{1}{2\eta}\frac{dp}{dx} z^2 + c_1 z + c_2$$

BCs: $u(0) = 0 \quad u(h) = u_0 \Rightarrow c_2 = 0$

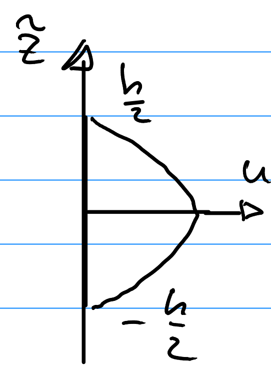
$$\boxed{u(z) = \frac{1}{2\eta}\frac{dp}{dx}\left(z^2 - hz\right) + \frac{u_0}{h} z} \quad (*)$$

Special cases



1) $\frac{dp}{dx} = 0 \rightarrow u(z) = \frac{u_0}{h} z$

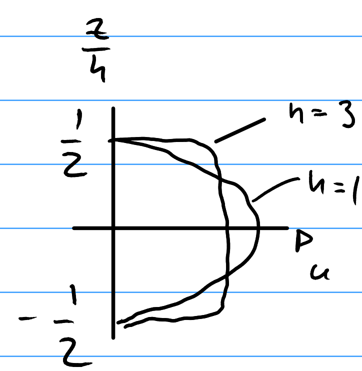
"Couette flow"



2) $u_0 = 0 \longrightarrow u(\tilde{z}) = \frac{1}{2\eta}\frac{dp}{dx}\left(\tilde{z}^2 - \frac{h^2}{4}\right)$

"Hagen-Poiseuille flow" $\quad \tilde{z} = z - \frac{h}{2}$

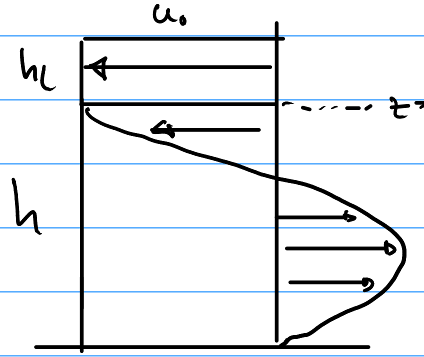
$$\bar{u} = \frac{h^2}{12\eta}\frac{dp}{dx}$$

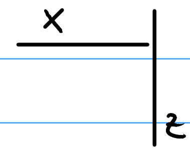


For powerlaw $\quad \dot{\varepsilon}_{xz} = C \cdot \tau^n$ (n odd)

$$u = \frac{C}{n+1}\left(\frac{dp}{dx}\right)^n\left(\tilde{z}^{n+1} - \left(\frac{h}{2}\right)^{n+1}\right)$$

Asthenospheric counterflow





$$u(h) = 0$$
$$u\ (h < 0) = u_0$$

Conservation of mass requires

$$u_0 h_L + \int_0^h u(z)\,dz = 0$$

Appropriate version of (*)

$$u = \frac{1}{2\eta}\frac{dp}{dx}\left(z^2 - hz\right) - \frac{u_0 z}{h} + u_0$$

insert and integrate

$$u_0 h_L + \left[\frac{1}{6\eta}\frac{dp}{dx}\left(z^3 - \frac{h}{2}z^2\right) - \frac{u_0}{2h}z^2 + u_0 z\right]_0^h = 0$$

and solve for pressure (Turcotte & Schubert 6-3)

$$(\#)\quad \frac{dp}{dx} = \frac{12\eta u_0}{h^2}\left(\frac{h_L}{h} + \frac{1}{2}\right)$$

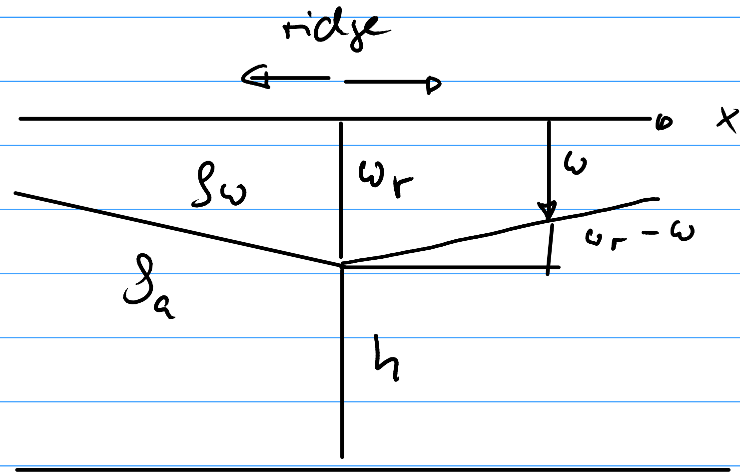
insert into (*)

$$u = u_0\left[1 - \frac{z}{h} + 6\left(\frac{h_L}{h} + \frac{1}{2}\right)\left(\frac{z^2}{h^2} - \frac{z}{h}\right)\right] \quad (\neq f(\eta))$$

Drag force on lithosphere at $z=0$

$$\tau = -\frac{2\eta u_0}{h}\left(2 + 3\frac{h_L}{h}\right) = 2.2\,\mathrm{MPa} \left\{ \begin{array}{l} \eta = 10^{19}\,\mathrm{Pa\,s} \\ u_0 = 5\,\mathrm{cm/yr} \\ h = 200\,\mathrm{km} \\ h_L = 100\,\mathrm{km} \end{array} \right.$$

Note that pressure increases with the direction of lithospheric motion, i.e. away from ridge. This has to be balanced by dynamic topography from isostasy



$$p = \rho_w g w + \rho_a g (h + w_r - w)$$

$$\frac{dp}{dx} = -(\rho_a - \rho_w) g \frac{dw}{dx} \rightarrow \frac{dp}{dx} > 0 \rightsquigarrow \frac{dw}{dx} < 0$$

decrease in ocean depth for increase in pressure

Insert into (#)

$$\frac{dw}{dx} = - \frac{12 \eta u_0}{(\rho_a - \rho_w) g h^2} \left(\frac{h_c}{h} + \frac{1}{2} \right)$$

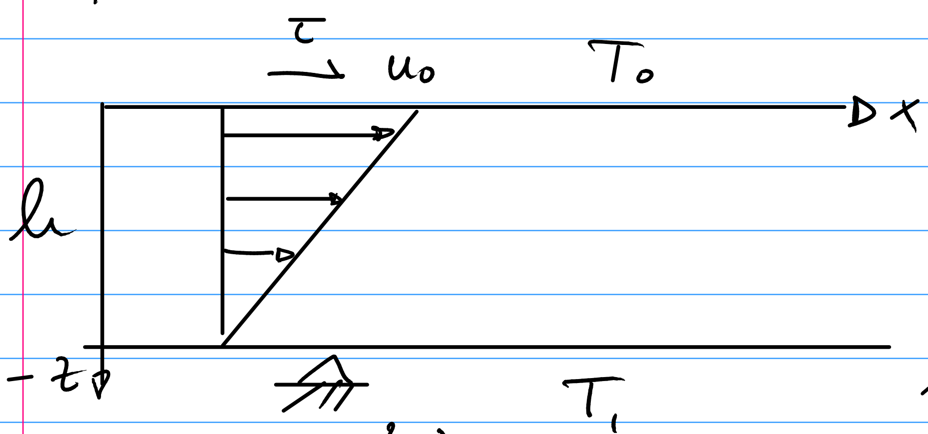
For $\rho_w = 1000\,kg/m^3$ and $\rho_a = 3300\,kg/m^3$, $g = 10\,\frac{m}{s^2}$

$\rightsquigarrow \quad \frac{dw}{dx} = -7.2 \cdot 10^{-4}$ for Pacific with $x = 10{,}000\,km$

expect a decrease in ocean depth of 7.2 km, which is not observed, indicating this model does not apply in general.

Shear heating in channel flow

So far, we've only looked at internal heating and advection/conduction for the energy eq., what about viscous heating/dissipation?



- fixed temperature at top/bottom
- constant stress at top

no-slip

$$u = u_0 \left(1 - \frac{z}{h}\right)$$

$$\tau = \eta \frac{du}{d(-z)} = \eta \frac{u_0}{h} = \tau_0$$

viscous dissipation $\quad \tau \cdot \dot{\epsilon} = \tau_0 \cdot \frac{u_0}{h} = \eta \frac{u_0^2}{h^2}$

rate of shear heating per volume $= H$

$$\frac{\partial^2 T}{\partial z^2} = \frac{H}{k} = -\eta \frac{u_0^2}{h^2 k}$$

Can solve for $\Theta = \frac{T - T_0}{T_1 - T_0} = \frac{T - T_0}{\Delta T}$

$$\Theta = \frac{z}{h}\left(1 + \frac{Pr\,E}{2}\right) - \frac{z^2}{h^2}\left(\frac{Pr\,E}{2}\right)$$

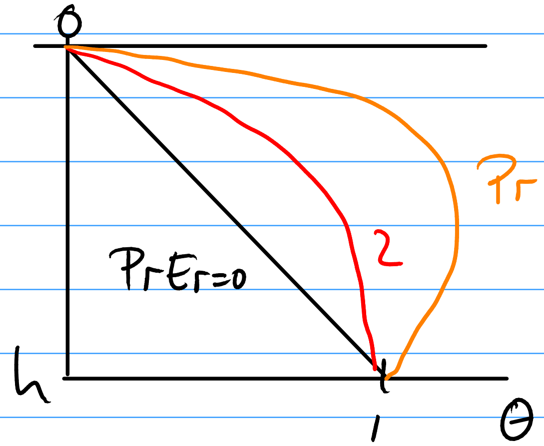
with the Prandtl number

$$Pr = \frac{\nu}{\varkappa} = \frac{\eta}{\rho \varkappa}$$

and the Eckert number

$$Er = \frac{u_0^2}{c_p \, \Delta T}$$

which measure kinetic to thermal energy content.

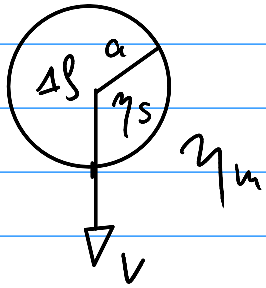


$$\theta_e = \theta - \theta_{cond} \quad ; \quad \theta_e^{max} = \frac{Pr\,E}{8}$$

With $\eta = 4 \cdot 10^{19}$ Pas, $u_0 = 5$ cm/yr, $k = 4\,W m^{-1} K^{-1}$, $\Delta T = 300°K$

$$\frac{Pr\,E}{2} = \frac{\eta \, u_0^2 / 2k}{\Delta T} \simeq 0.04 \rightarrow \theta_e^{max} \sim 1\%$$

Stokes flow (Consider density driven flow)



$$F_B = \frac{4\pi}{3} a^3 \Delta\rho g \quad ; \eta_s = \infty$$

$$F_D \propto 4\pi a^2 \eta_m \frac{v}{a}$$

$$\rightsquigarrow \quad v \propto \frac{\Delta\rho \, a^2 g}{3 \eta_m}$$

(Turcotte & Schubert, 6-14)

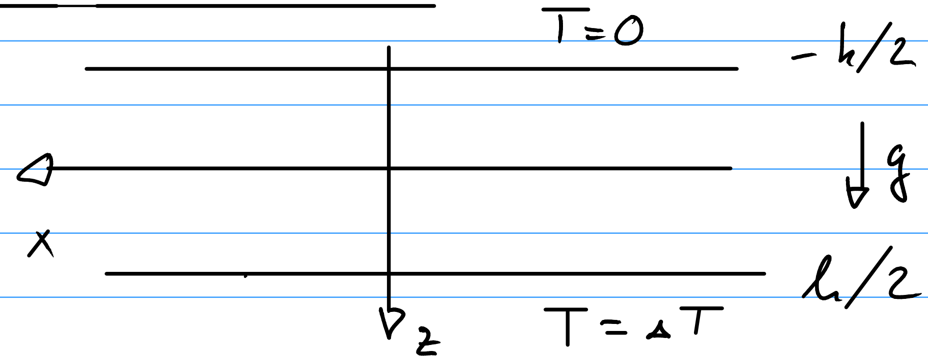
$$v = C \cdot \frac{\Delta\rho g a^2}{\eta_m}$$

$$C = \frac{2 + 2\eta'}{6 + 9\eta'} \text{ with } \eta' = \frac{\eta_s}{\eta_m} \text{ i.e. } C = \frac{2}{9} \text{ for } \eta' \to \infty$$

$$C = \frac{1}{3} \text{ for } \eta' \to 0$$

Thermal convection

Boundary layer instability for the Rayleigh-Benard problem



Fluid layer of thickness h heated from below and cooled from above.

Energy equation (without viscous or internal heating)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{u}\cdot\underline{\nabla} T = \varkappa \nabla^2 T$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial z} = \varkappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

In initial steady state, in the absence of convection (ΔT small),

$$\underline{V} = 0 \quad ; \quad \frac{\partial T}{\partial t} = 0 \quad ; \quad \frac{\partial T}{\partial x} = 0$$

conductive state $\quad T_c = \Delta T\left(\frac{1}{2} + \frac{z}{h}\right)$ primed quantities denote deviation from conductive

deviations $\quad T' = T - T_c$ small

For onset of convection, u', v' small & insert in energy eq.

$$\partial_t T' + u'\partial_x T' + v'\partial_z T' + \frac{v'\Delta T}{h} = \varkappa\left(\partial_{xx} T' + \partial_{zz} T'\right)$$

Neglect products of two small quantities (non-linear terms)

$$\partial_t T' + \frac{v'}{h} \Delta T = \varkappa \left(\partial_{xx} T' + \partial_{zz} T' \right)$$

(linearized stability analysis, valid for small amplitudes)

Continuity eq. $\quad \partial_x u' + \partial_z v' = 0$

Momentum eq. $\quad -\partial_x p' + \eta \left(\partial_{xx} u' + \partial_{zz} u' \right) = 0$

$$-\partial_y p' + \eta \left(\partial_{xx} v' + \partial_{zz} v' \right) = \rho \alpha g T'$$

with $\quad \Delta \rho = \rho_0 \alpha \Delta T$ (Boussinesq approximation)

BCs $\quad T' = v' = 0 \quad$ for $z = \pm \frac{h}{2}$

For no-slip $\quad u' = 0 \quad$ for $z = \pm \frac{h}{2}$

free-slip $\quad \tau'_{xz} = 0 \quad$ for $z = \pm \frac{h}{2}$

or $\partial_z u' = 0 \quad$ for $z = \pm \frac{h}{2}$

Assume free-slip and introduce stream function

$$u' = -\partial_z \Psi' \qquad v' = \partial_x \Psi'$$

Plug into force balance eqn.

$$-\partial_x p' - \eta\left(\partial_{xxz}\Psi' + \partial_{zzz}\Psi'\right) = 0 \qquad (1)$$

$$\partial_z p' + \eta\left(\partial_{xxx}\Psi' + \partial_{zzx}\Psi'\right) = \rho g \alpha T' \qquad (2)$$

$$\partial_t T' + \frac{\Delta T}{h}\partial_x \Psi' = \kappa\left(\partial_{xx}T' + \partial_{zz}T'\right)$$

eliminate p' from (1) and (2)

$$\eta\left(\partial_{xxxx}\Psi' + 2\partial_{xxzz}\Psi' + \partial_{zzzz}\Psi'\right) = \rho g \alpha\, \partial_x T' \qquad (*)$$

Solve these coupled, linear PDEs by separation of variables and assume solutions are of form

$$\Psi' = \Psi_0' \cos\left(\frac{\pi z}{h}\right)\sin\left(\frac{2\pi x}{\lambda}\right)e^{\phi t}$$

$$T' = T_0' \cos\left(\frac{\pi z}{h}\right)\cos\left(\frac{2\pi x}{\lambda}\right)e^{\phi t} \qquad (\#)$$

These functions automatically satisfy the BCs and correspond to horizontally periodic perturbations of the conductive state. ϕ determines the growth rate, $\phi > 0$ fluctuations will grow, $\phi < 0$ will die out.

Insert (#) into (*)

$$\left(\phi + \kappa\left(\frac{\pi^2}{h^2} + \frac{4\pi^2}{\lambda^2}\right)\right)T_0' = -\frac{2\pi\Delta T}{\lambda h}\Psi_0'$$

$$\eta\left(\frac{4\pi^2}{\lambda^2} + \frac{\pi^2}{h^2}\right)^2\Psi_0' = -\frac{2\pi}{\lambda}\rho g \alpha T_0'.$$

Can eliminate T_0' and Ψ_0' to get

$$\phi' = \frac{\phi}{t_c} = \frac{Ra\, k^2 - (\pi^2 + k^2)^3}{(\pi^2 + k^2)^2}$$

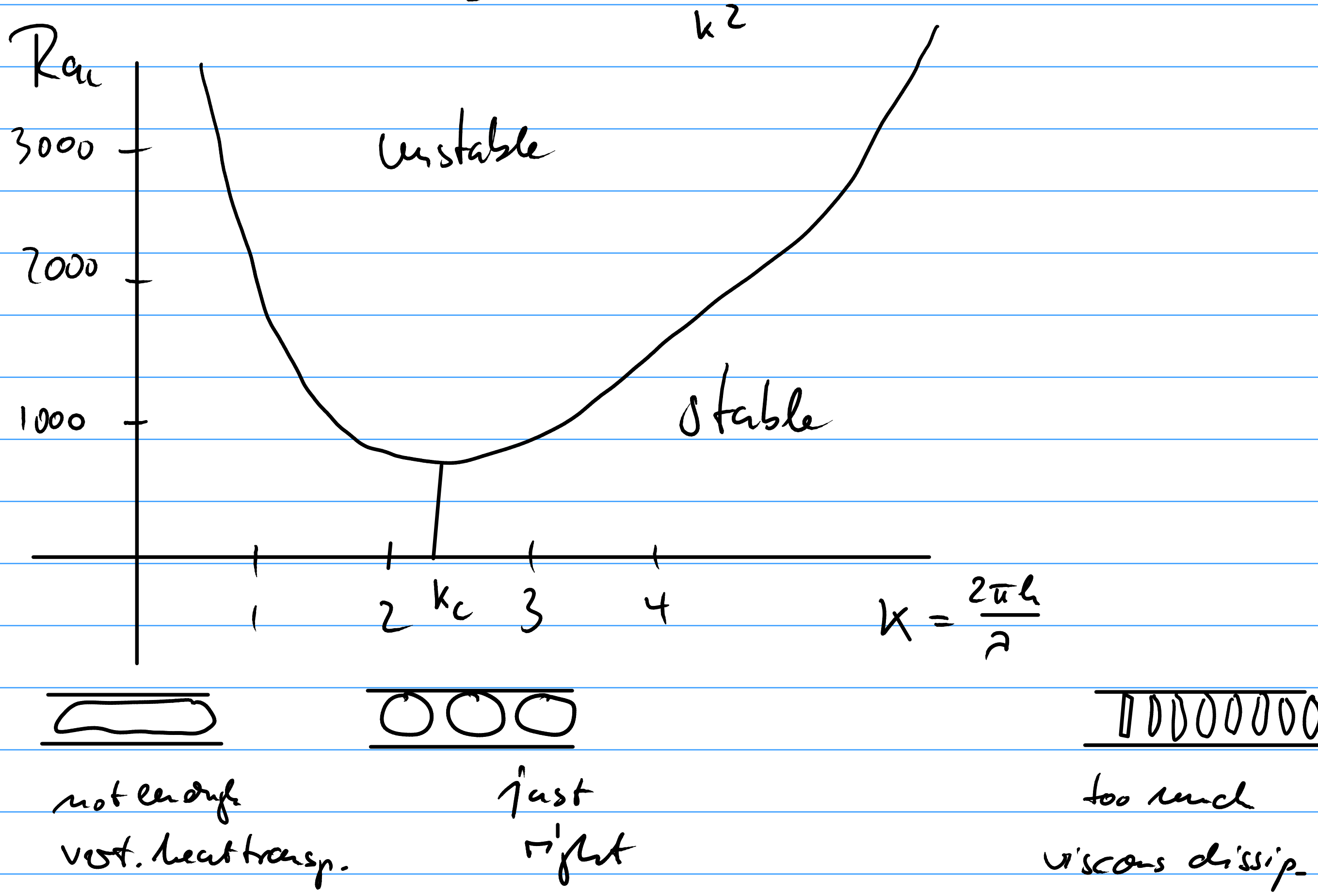
With characteristic time $t_c = \frac{h^2}{\kappa}$

non-dimensional wave number $k = \frac{2\pi}{\lambda'} = \frac{2\pi h}{\lambda}$

and the Rayleigh # $= \frac{\Delta T g \alpha \rho h^3}{\eta \kappa}$

$\phi' > 0$ and perturbations unstable for

$$Ra > Ra_c = \frac{(\pi^2 + k^2)^3}{k^2}$$



$\partial_k Ra_c = 0 \rightsquigarrow k_c = \frac{\pi}{\sqrt{2}}$

$$\left.\begin{aligned} \lambda_c &= 2\sqrt{2}\,h = 2.83h \\ Ra_c(\lambda_c) &= 657.5 \end{aligned}\right\} \text{for free slip}$$

$$\left.\begin{aligned} \lambda_c &= 2.016h \\ Ra_c(\lambda_c) &= 1707.8 \end{aligned}\right\} \text{for no-slip}$$

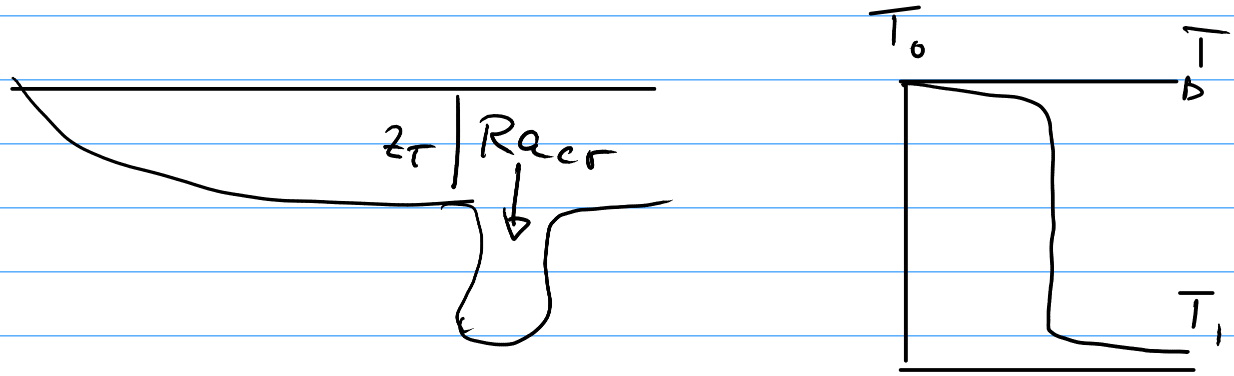
Can also consider internally heated medium, cooled from above, insulating below with heating rate H' per unit mass ($[H'] = W/kg$)

$$Ra_H = \frac{\rho^2 \alpha g H' h^5}{k \eta \varkappa} = \frac{\rho g \alpha H h^5}{k \eta \varkappa}$$

$$\left.\begin{aligned} \lambda_c &= 3.5h \\ Ra_c(\lambda_c) &= 867.8 \end{aligned}\right\} \text{free-slip}$$

$$\left.\begin{aligned} \lambda_c &= 2.38h \\ Ra_c(\lambda_c) &= 2772 \end{aligned}\right\} \text{no-slip}$$

Boundary layer instability (transient solutions)



critical Rayleigh number for the local condition in the upper thermal boundary layer

$$Ra\left(\Delta T = \frac{T_1 - T_0}{2}, h = z_L = 2.32\sqrt{\varkappa t}\right)$$

$$= \frac{\rho g \alpha (T_1 - T_0) z_L^3}{2 \eta \varkappa} = Ra_c \simeq 657.5$$

for free slip

(onset of small scale convection).

Can solve for t_c at which z_L is large enough, can integrate half space cooling for mean heat flux up to t_c

$$\bar{q} = \frac{k(T_1 - T_0)}{\sqrt{\pi \varkappa t_c}}$$

If we define the Nusselt #

$$Nu = \frac{\bar{q}}{q_{cond}} = \frac{\bar{q} \cdot h}{k(T_1 - T_0)}$$

then $$\boxed{Nu = 1.04\left(\frac{Ra}{Ra_{cr}}\right)^{\frac{1}{3}} = 0.12\, Ra^{\frac{1}{3}}}$$

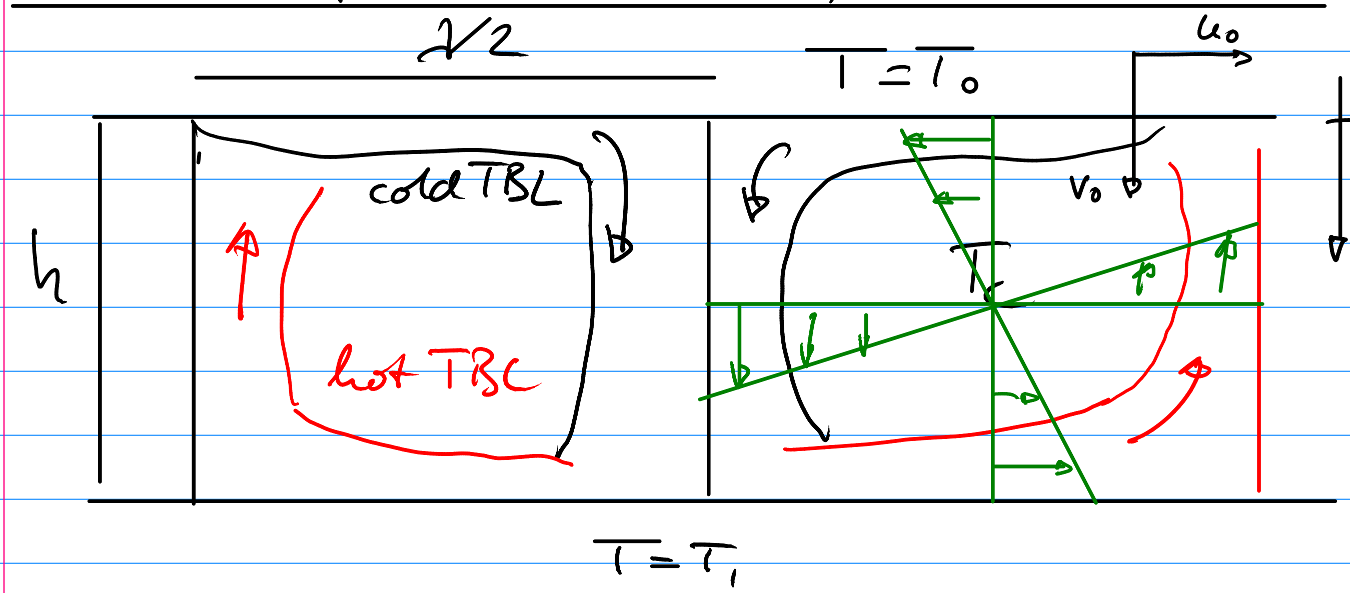
$Ra_c = 657.5$

For typical values (see Turcotte and Schubert, 6-20) can find

$$t_c = \frac{22.3\, h^2}{\kappa\, Ra^{2/3}}$$

~ 50.5 Myr for $h = 700$ km

Finite amplitude boundary layer model



Estimate the downward pull of the cold thermal boundary layer akin to the problem set on plate driving forces (see T&S, 6-21)

$$F_B = 2\rho_0 g\alpha h (T_c - T_0) \frac{u_0}{v_0} \sqrt{\frac{\varkappa \lambda}{2\pi u_0}}$$

ratio of horizontal to vertical velocities

↳ timescale for TBL formation

Assume linear velocity profile, for simplicity, where conservation of mass (incompressibility) gives

$$\frac{v_0 \lambda}{2} = u_0 h$$

(velocities do not satisfy BCs, mostly approximations of core flow). Rate of viscous shearing work on vertical boundaries is

$$F_{vv} = h\, v_0\, \tau_{cv} = h v_0 \left(\frac{4 v_0}{\lambda} \eta\right)$$

on horizontal boundaries

$$F_{vh} = \frac{\partial}{2} u_0 \tau_{ch} = \frac{\partial u_0}{2} \left(\frac{2u_0}{h} \eta \right)$$

total buoyancy force = total viscous force

$$2 F_B = 2 F_{vv} + 2 F_{vh}$$

$$\rightsquigarrow \quad u_0 = \frac{\kappa}{h} \frac{\alpha^{7/3}}{(1+\alpha^4)^{2/3}} \left(\frac{Ra}{2\sqrt{\pi}} \right)^{\frac{2}{3}} \qquad (*)$$

$$\alpha = \frac{\partial}{2h}$$

Can use (*) and $Q = 2k(T_c - T_0) \left(\frac{u_0 \partial}{2\pi \kappa} \right)^{\frac{1}{2}}$

to get

$$Nu = \frac{1}{2^{1/3} \pi^{2/3}} \frac{\alpha^{2/3}}{(1+\alpha^4)^{1/3}} Ra^{1/3}$$

Aspect ratio $\alpha = \frac{\partial}{2h}$ is a free parameter. From perturbation analysis, would expect $\alpha = \sqrt{2}$. For finite amplitude, require maximum Nu

$$\frac{\partial Nu}{\partial \alpha}(\alpha_m) = 0 \quad \rightsquigarrow \alpha_m = 1.$$

With this α, find

$$u_0 = 0.271 \frac{\kappa}{h} Ra^{2/3}$$

$$Nu = 0.294 \, Ra^{1/3}$$

$$v_c = \frac{\kappa}{h} \qquad \bar{q} = \frac{k(T_1 - T_0)}{h} \cdot Nu$$

For internal heating, can also determine α_m to find $a_m = 1$, as for bottom heating; then

$$u_0^i = 0.354 \frac{\kappa}{h} Ra_H^{1/2}$$